Is quantum field theory complex enough?

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University of Illinois, Chicago

February 28, 2018

Outline

complex fields arise in QFT naturally

- Part 1 - Semi-classical expansion
  and the necessity of complex saddles

- Part 2 - Strongly coupled systems:
  the notorious sign problem and complex fields to rescue!

Part 1

Semi-classical physics and the necessity of complex saddles

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Semi-classical physics and the necessity of complex saddles

Perturbation theory

perturbation expansions in physics are generically divergent

$$\mathcal{O}(g) \sim \sum_{n=0}^{\infty} c_n g^n \qquad c_n \sim n!$$

Stark effect, Euler-Heisenberg, string theory (Shenker), QFT in curved backgrounds, derivative expansions, hydrodynamics (GB et. al, 2015). . .

Divergence of Perturbation Theory in Quantum Electrodynamics

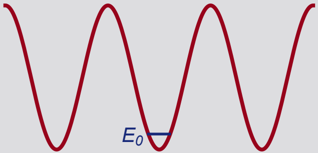
F. J. Dyson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge.

Perturbation theory

divergence $\Leftrightarrow$ non-perturbative physics

e.g. 1d quantum mechanics with cosine potential:



$$-\frac{g^2}{2}\frac{d^2\psi}{d\phi^2} + \cos(\phi)\psi = E\psi$$

Ground state energy: $E_0(g) \sim \sum_n c_n g^n \quad , \quad c_n \sim \frac{n!}{(2S)^n}$

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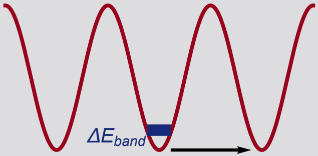

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non-perturbative physics (tunneling):

- band width: $\Delta E_{band} \sim e^{-\frac{1}{g}S}$

Perturbation theory

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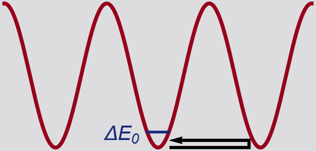

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non-perturbative physics (tunneling):

- band width: $\Delta E_{band} \sim e^{-\frac{1}{g}S}$
- correction to E_0: $\Delta E_0 \sim e^{-\frac{2}{g}S}$

Semi-classical expansion

2 lessons:

I - perturbation theory is incomplete, has to be "enhanced" by non-perturbative physics

$$E_0(g) \sim \underbrace{\sum_{n=0}^{\infty} c_n g^n}_{perturbative} + \sum_{k=0}^{\infty} \underbrace{\left(e^{\frac{-2S}{g}}\right)^k}_{multiple-tunneling} \times \underbrace{\sum_{n=0}^{\infty} c_n^{(k)} g^n}_{fluctuations}$$

Semi-classical expansion

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II - perturbative and non-perturbative physics are related (i.e. $c_n \Leftrightarrow c_n^{(k)}$)

(GB, Dunne, Ünsal et. al, for the previous example see also, GB et. al; JHEP 1705 (2017) 087)

Semi-classics: the path integral

semi-classical expansion follows from the path integral

$$Z = \int [D\phi] e^{-\frac{1}{g}S[\phi]} \approx \sum_{k \in saddles} e^{-\frac{1}{g}S_k} \times fluct_k(g)$$

non-perturbative physics

$\Updownarrow$

saddles

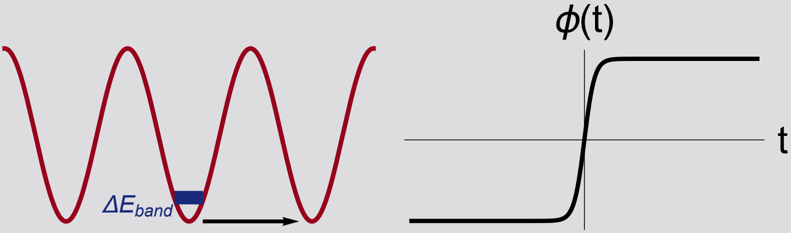
solutions of the Euclidean equations of motion ($\frac{\delta S}{\delta \phi} = 0$)

+ fluctuations

Semi-classics: the path integral

$$Z = \int [D\phi] e^{-\frac{1}{g} S[\phi]} \approx \sum_{k \in saddles} e^{-\frac{1}{g} S_k} \times fluct_k(g)$$

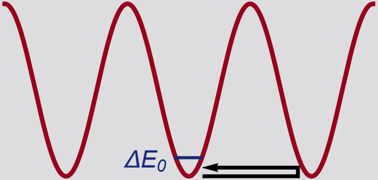
k = 1-instanton (action=S) ⇔ leading band width

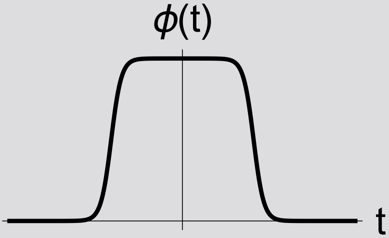


Semi-classics: the path integral

$$Z = \int [D\phi] e^{-\frac{1}{g}S[\phi]} \approx \sum_{k \in saddles} e^{-\frac{1}{g}S_k} \times fluct_k(g)$$

k = 2-instanton (action=2S) $\Leftrightarrow$ non-perturbative correction to E_0





Exact semi-classics

this can be made into an exact statement

$$Z = \sum_{k \in saddles} n_k \times e^{-\frac{1}{g} S_k} \times \int_{\mathcal{J}_k} [D\phi] e^{-\frac{1}{g}(S[\phi] - S_k)}$$

all-orders fluctuations around saddle k

$\Updownarrow$

$\mathcal{J}_k$: a particular contribution to Z.

a.k.a. "Lefschetz thimble":

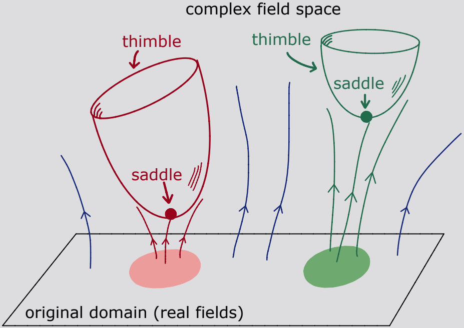
exact steepest descent contour for the path integral

Exact semi-classics: finding saddles

follow an "equation of motion": flow

$$\frac{d\phi_a}{d\tau} = \overline{\frac{\partial S}{\partial \phi_a}} \qquad \phi_a = x_a + iy_a \begin{cases} \dfrac{dx_a}{d\tau} = \dfrac{\partial S_R}{\partial x_a} = \dfrac{\partial S_I}{\partial y_a} \\ \dfrac{dy_a}{d\tau} = \dfrac{\partial S_R}{\partial y_a} = -\dfrac{\partial S_I}{\partial x_a} \end{cases}$$

- saddle points are fixed points of flow
- $|e^{-S}| = e^{-S_R}$ decreases with flow (except for saddles)



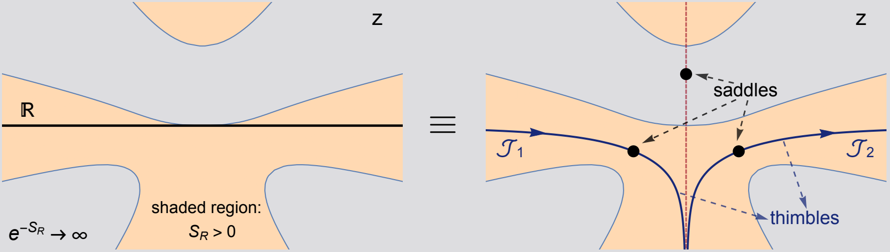
flow time $\rightarrow \infty$

original path integration domain $\rightarrow$ thimbles (saddles + fluctuations)

Path integral and topology

Cauchy's theorem

$$\mathcal{Z} = \int_{\mathbb{R}} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)} = \int_{\mathcal{J}_1 + \mathcal{J}_2} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)}$$



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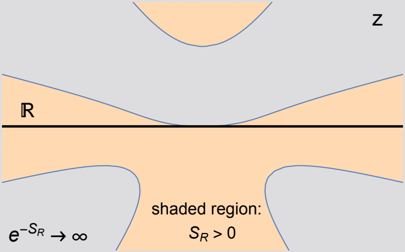

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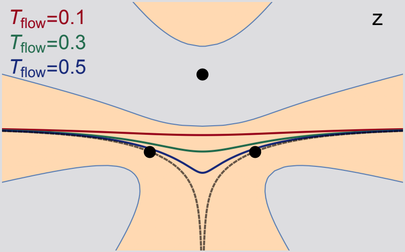
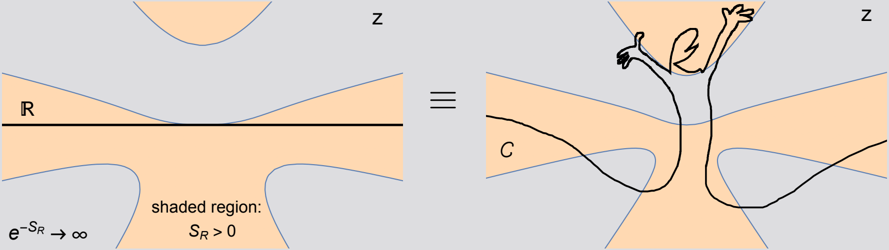


Path integral and topology

Cauchy's theorem

$$\mathcal{Z} = \int_{\mathbb{R}} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)} = \int_{\mathcal{C}} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)}$$



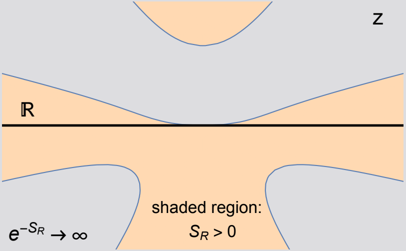
integral only depends on asymptotic directions

"Picard-Lefschetz theory" (Pham, Fedoryuk, Witten ...)

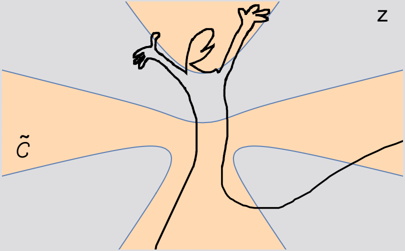
Path integral and topology

Cauchy's theorem

$$\mathcal{Z} = \int_{\mathbb{R}} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)} \neq \int_{\tilde{\mathcal{C}}} dz\, e^{-i\left(\alpha z^2 + \frac{1}{4} z^4\right)}$$



$\neq$



integral only depends on asymptotic directions

"Picard-Lefschetz theory" (Pham, Fedoryuk, Witten ...)

Semi-classics: complex saddles

in the thimble decomposition, some of the saddles are *complex* and still contribute to semiclassical expansion even when the original domain is real fields

this is not by choice, it is dictated by the path integral

they contain important physics

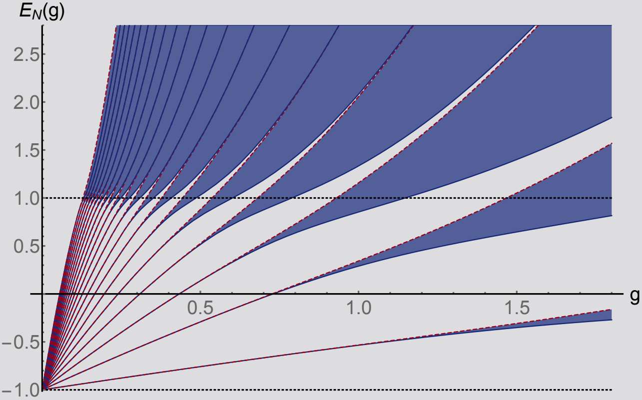
Part 1

Semi-classical physics and the necessity of complex saddles

back to our example: high energy limit

$\approx$ free particle on a circle

$\Downarrow$

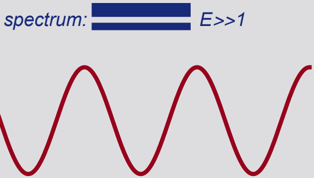


$\Uparrow$

tight binding, asymptotic series, band width $\sim e^{-\frac{S}{g}}$

Semi-classical high energy limit

$$N \to \infty,\ g << 1,\ Ng >> 1$$

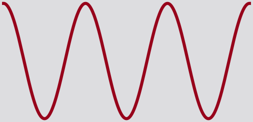


perturbative expansion:

$$E_{pt} \sim \frac{g^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{g}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{g}\right)^8 + \dots\right)$$

at any finite N, E_{pt} has poles (at order $1/g^{4N}$)

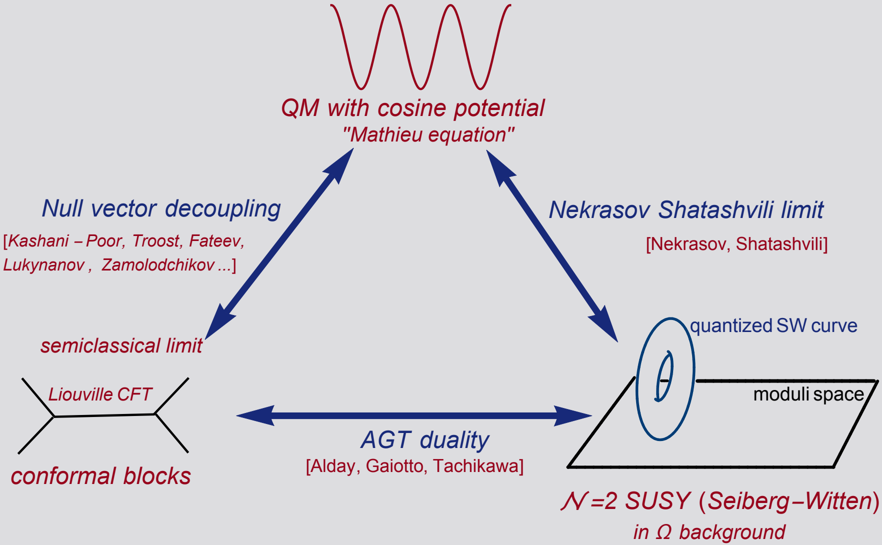
perturbative expansion can't be the whole story...



$$-\frac{g^2}{2}\frac{d^2\psi}{d\phi^2} + \cos(\phi)\psi = E\psi$$

hidden in this quantum mechanical problem, there is a very rich quantum field theory structure

Dualities



QM with cosine potential
"Mathieu equation"
Null vector decoupling
[Kashani – Poor, Troost, Fateev,
Lukynanov , Zamolodchikov ...]
Nekrasov Shatashvili limit
[Nekrasov, Shatashvili]
semiclassical limit
quantized SW curve
Liouville CFT
moduli space
conformal blocks
AGT duality
[Alday, Gaiotto, Tachikawa]
$\mathcal{N}$=2 SUSY (Seiberg–Witten)
in Ω background

Gauge theory puzzle

$E_{pt} \Leftrightarrow$ Nekrasov twisted superpotential

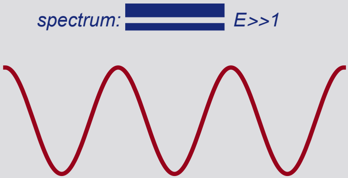
$$\mathcal{W}_{NS}^{inst.} = \frac{1}{4\pi i}\left(\frac{2\Lambda^4}{(4a^2-\epsilon_1^2)\,\epsilon_1^4} + \frac{\Lambda^8\left(20a^2+7\epsilon_1^2\right)}{4\left(a^2-\epsilon_1^2\right)\left(4a^2-\epsilon_1^2\right)^3} + \ldots\right)$$

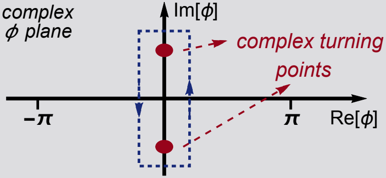
a=Ng/2 $\Updownarrow$ $\epsilon_1 = g, \epsilon_2 \to 0$

$$E_{pt} = \frac{i\pi}{2}\Lambda\frac{\partial\mathcal{W}_{NS}^{inst.}}{\partial\Lambda} = \frac{g^2}{8}\left(\frac{8\Lambda^4}{(N^2-1)\,g^4} + \frac{8\Lambda^8\left(5N^2+7\right)}{(N^2-4)\,(N^2-1)^3\,g^8} + \ldots\right)$$

same poles

Resolution in QM: Complex instantons





tunneling in complex plane:

complex instantons (GB, Dunne, JHEP 1502 (2015) 160)

non-perturbative gap splitting:

the "cure" for the poles

$$\Delta E_N^{\rm gap} \sim e^{-\frac{2\pi}{g} S_{inst}} \sim e^{-2N(\log(gN)-1)} \sim g^{-2N}$$

Gauge theory puzzle

$E_{pt} \Leftrightarrow$ Nekrasov twisted superpotential

$$E_{pt} = \frac{i\pi}{2}\Lambda\frac{\partial \mathcal{W}_{NS}^{inst.}}{\partial\Lambda} = \frac{g^2}{8}\left(\frac{8\Lambda^4}{(N^2-1)\,g^4} + \frac{8\Lambda^8\,(5N^2+7)}{(N^2-4)\,(N^2-1)^3\,g^8} + \dots\right)$$

- gauge theory realization of the complex instantons and perturbative $\leftrightarrow$ non-perturbative relations (GB, Dunne, Ünsal, 2015) ?
  monopole pair creation? (see also Gorsky et. al 2015)
- what about the conformal blocks? any relations to quasi normal modes in AdS_3 (J. Kaplan et. al 2016) ?

Complex instantons

(GB, Dunne, JHEP 1502 (2015) 160)

pair production in a monochromatic electric field

$$\mathcal{E}\cos(\omega t)$$

where $g \Leftrightarrow \frac{4\omega^2}{\mathcal{E}}$, $N \Leftrightarrow \frac{m}{\omega} \sim$ photon #

2 limits:

$\frac{m\omega}{\mathcal{E}} \ll 1 \leftrightarrow$ constant field $\leftrightarrow$ real instantons

$\frac{m\omega}{\mathcal{E}} \gg 1 \leftrightarrow$ multi-photon $\leftrightarrow$ complex instantons

$$rate = e^{-\frac{m^2\pi}{\mathcal{E}} f\left(\frac{m\omega}{\mathcal{E}}\right)} \sim \begin{cases} e^{-\frac{m^2\pi}{\mathcal{E}}} & \Leftrightarrow \quad \text{band width} \\ e^{-\frac{4m}{\omega}\log\left(4\frac{m\omega}{\mathcal{E}}\right)} & \Leftrightarrow \quad \text{gap width} \end{cases}$$

similar phenomenon in gauge theory?

monopole pair creation? (see also Gorsky et. al 2015)

Part 2

Strongly interacting systems:

real-time, finite density physics, the notorious sign problem, and flow to the rescue!

based on:

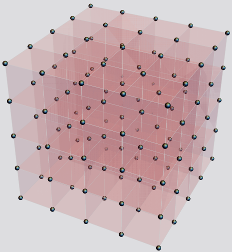
Phys. Rev. Lett. 117, 081602, Phys. Rev. D93, 014504, Phys. Rev. D93, 094514, Phys. Rev. D94, 045017, Phys. Rev. D95, 014502, Phys. Rev. D96, 034513, Phys. Rev. D95, 114501, JHEP 05 (2016) 053

with A. Alexandru, P. Bedaque, G. Ridgway*, S. Vartak**, N. Warrington***

* UMD undergraduate → MIT, ** UMD undergraduate → Yale, ***UMD graduate student

A crash course on lattice field theory

when there is no weak coupling expansion, we can just compute the whole thing...



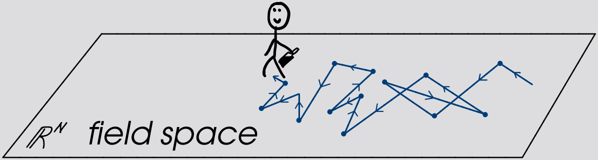
discrete space-time

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \underbrace{d\phi_1 \dots d\phi_N}_{\textit{finite}} \, e^{-S[\phi]} \mathcal{O}[\phi]$$

importance of field configuration $\phi \propto e^{-S[\phi]}$

A crash course on lattice field theory

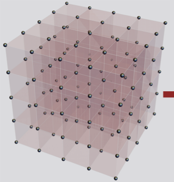
"importance sampling" (Monte-Carlo method):
pick out the important (small action) configurations



path integral ≈ statistical average with $P[\phi] \propto e^{-S[\phi]}$

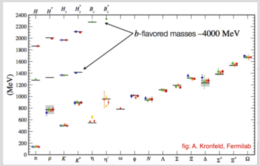
$$\langle \mathcal{O} \rangle \approx \frac{1}{\mathcal{N}} \sum_{a=1}^{\mathcal{N}} \mathcal{O}[\phi^{(a)}]$$

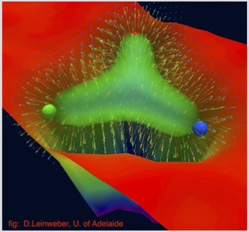
A crash course on lattice field theory

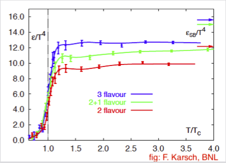


lattice

importance sampling (Monte-Carlo)





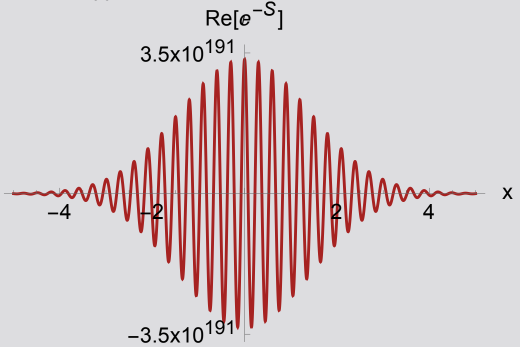


In many cases S, the (effective) action, is complex

- dynamical problems: *out-of-equilibrium, transport, equilibration, quantum chaos...*
- nonzero density: *dense quark matter, neutron stars, strongly correlated electronic systems, Hubbard model...*
- gauge theories with non-zero θ: *strong CP problem, axion*
- matrix models: *supergravity, string theory...*

when the action is complex...

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x+42i)^2}\, dx = 2\sqrt{\pi}$$



“the sign problem"

The sign problem

importance $\propto e^{-S_R}$: *"reweighting"*

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O}\, e^{-iS_I} \rangle_{S_R}}{\langle e^{-iS_I} \rangle_{S_R}}$$

- $\langle e^{-iS_I} \rangle_{S_R} \propto e^{-volume/T}$
- need exponentially large resources

The sign problem

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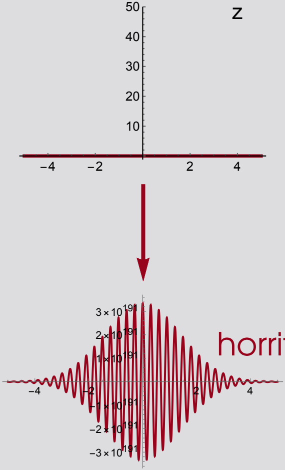
The sign problem

Flow to the rescue

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}(z+42i)^2}\ dz = 2\sqrt{\pi}$$



horrific sign problem

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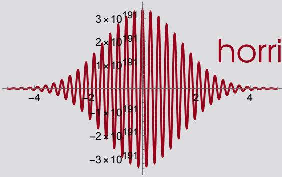
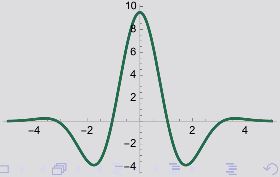


horrific sign problem

mild sign problem



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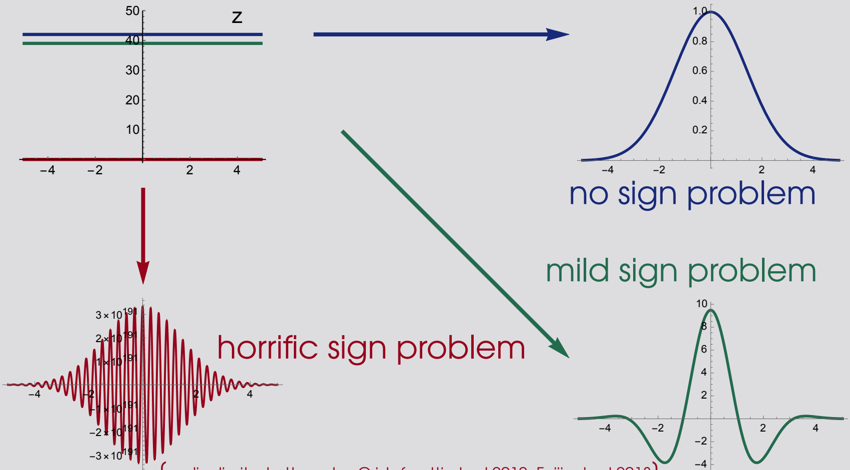
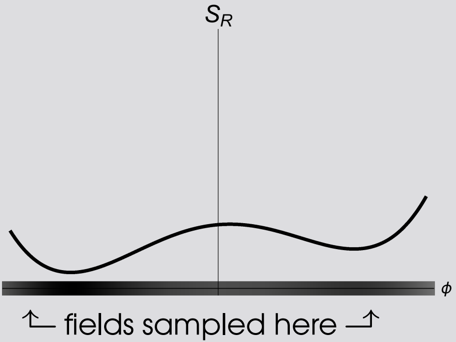


(earlier limited attempts : Cristoforetti et. al 2012, Fujii, et. al 2013)

Flow to the rescue

flow $\rightarrow$ S_R increases, S_I remains the same

$$\frac{d\phi_a}{d\tau} = \overline{\frac{\partial S}{\partial \phi_a}} \qquad \phi_a = x_a + i y_a \begin{cases} \frac{dx_a}{d\tau} = \frac{\partial S_R}{\partial x_a} = \frac{\partial S_I}{\partial y_a} \\ \frac{dy_a}{d\tau} = \frac{\partial S_R}{\partial y_a} = -\frac{\partial S_I}{\partial x_a} \end{cases}$$



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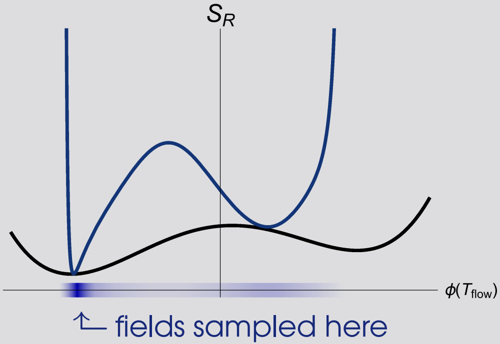
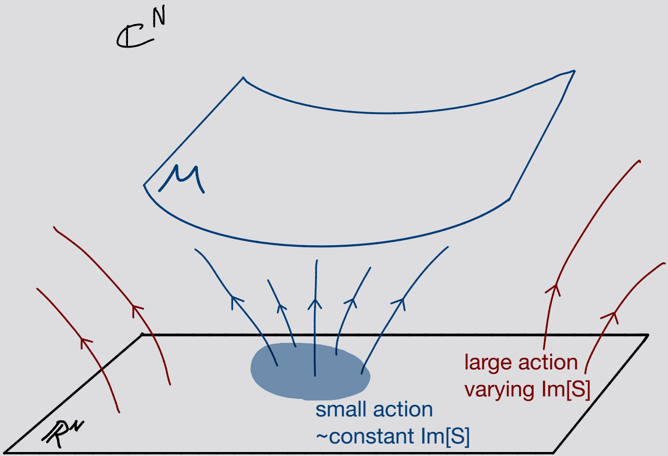


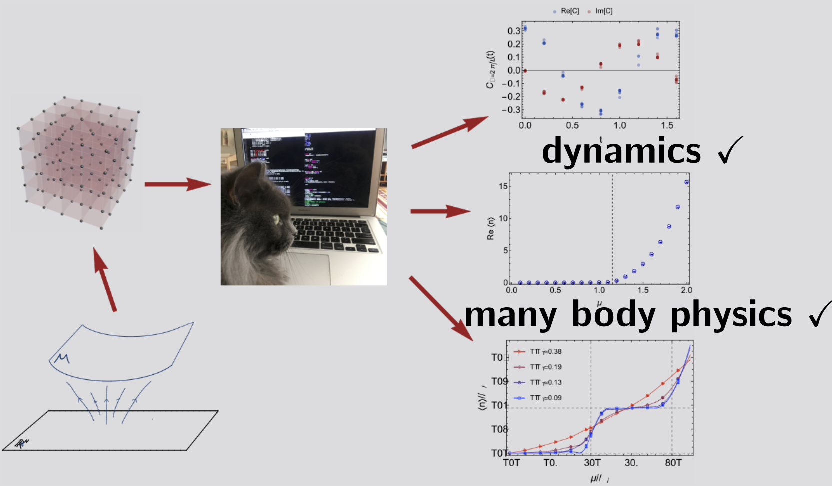
Flow to the rescue



- path integral on $\mathcal{M}$ = path integral on $\mathbb{R}^N$
- sign problem $\mathcal{M}$ << sign problem on $\mathbb{R}^N$

Importance sampling on $\mathcal{M}$

(GB et. al; PRD 93, 014504, JHEP 05 (2016) 053, PRD 93, 094514)



Our Results

quantum dynamics

- anharmonic oscillator
  (Phys. Rev. Lett. 117, 081602)
- 1+1d interacting Bose gas
  (Phys. Rev. D. 95, 114501)

many body systems

- 1d Hubbard model
  (PRD 93 014504, JHEP 1605 053)
- 2d Thirring model
  (Phys. Rev. D95, 014502)
- 4d interacting Bose gas
  (Phys. Rev. D94, 045017)

Quantum dynamics: out-of-equilibrium, transport

$$\begin{aligned} \langle \mathcal{O}(t)\mathcal{O}(0)\rangle &= Tr[e^{-\beta \hat{H}} \mathcal{O}(t)\mathcal{O}(0)] \\ &= \frac{1}{Z}\int [d\phi]\, e^{\frac{i}{\hbar}S[\phi]}\, \mathcal{O}(t)\mathcal{O}(0) \end{aligned}$$

equilibration, viscosity, conductivity...

$e^{\frac{i}{\hbar}S[\phi]}$ leads to quantum interference

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and the ultimate sign problem!

$$\langle e^{-iS_I}\rangle_{S_R} = 0$$

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equilibration, viscosity, conductivity...

$e^{\frac{i}{\hbar}S[\phi]}$ leads to quantum interference

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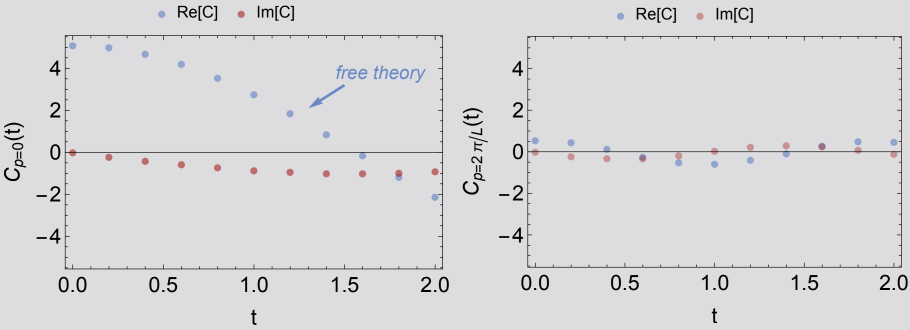
let's solve it...

Dynamics: 1+1d quantum field theory

(GB et. al, Phys. Rev. D95, 014502, 2017)

interacting Bose gas: $\mathcal{L}_I = \frac{\lambda}{4!}\phi^4$

weak coupling: $\lambda = 0.1$



$$C_p(t) = \langle \phi(t, p)\phi(0, p)\rangle_\beta$$

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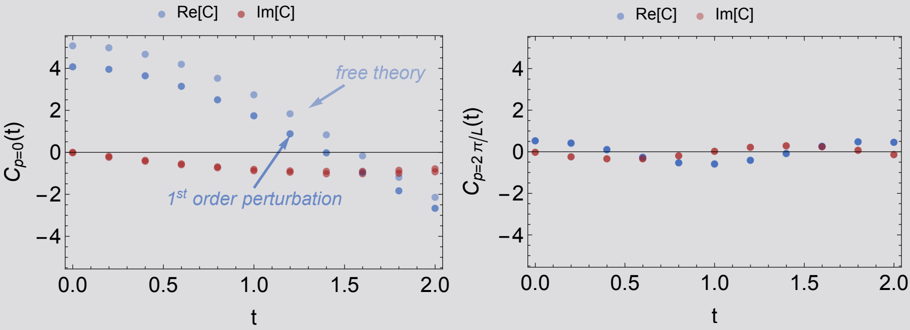

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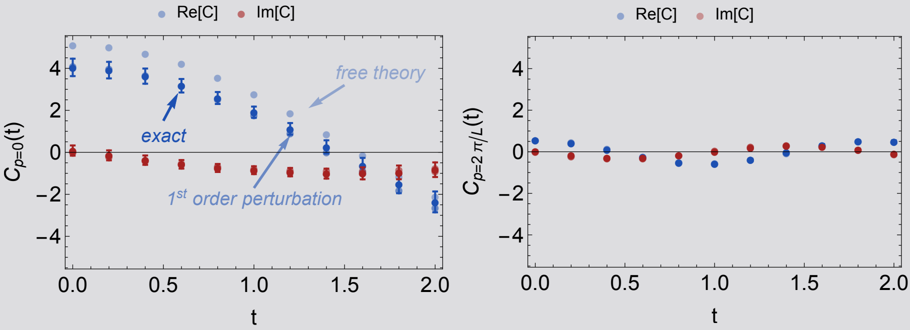

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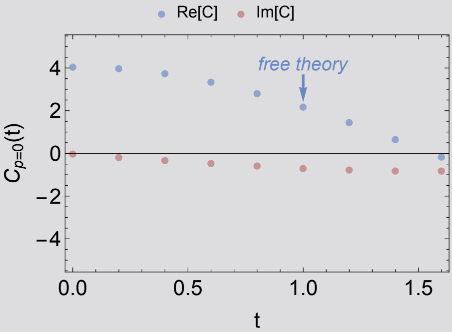

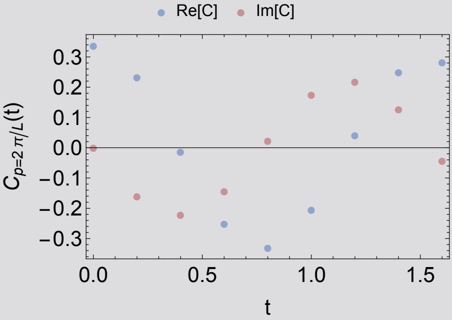

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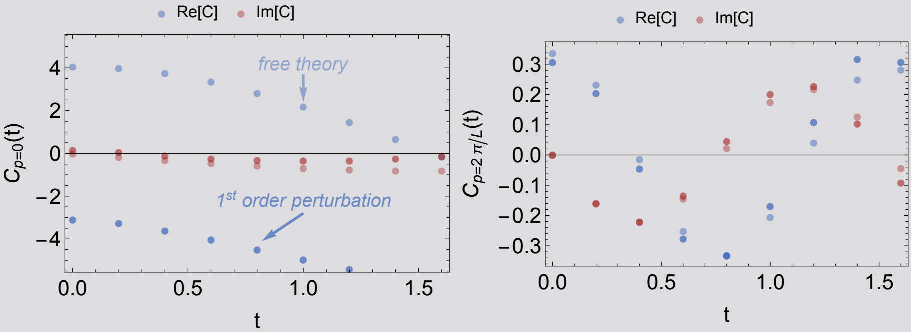

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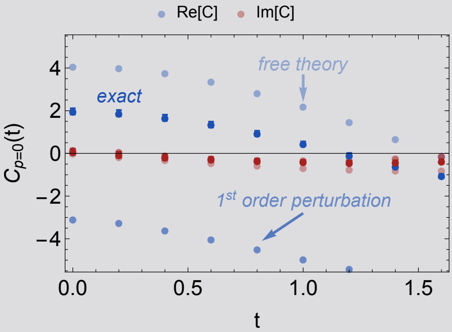

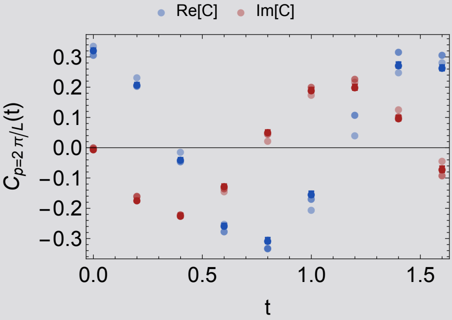

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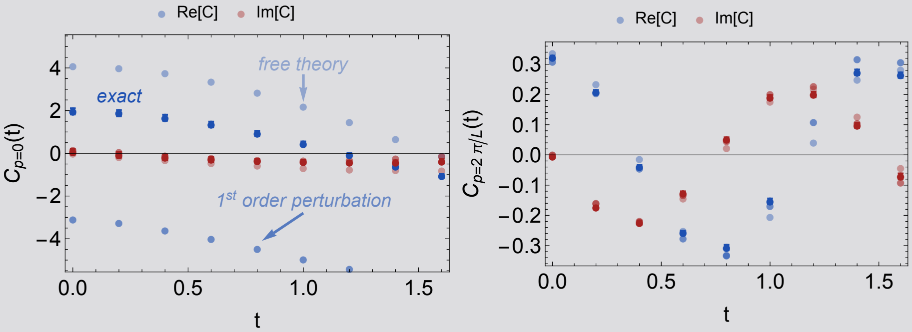
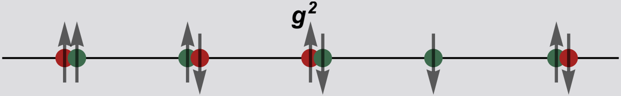


first direct lattice computation

Many body physics: 2d Thirring model

(GB et. al, Phys. Rev. D95, 014502, 2016)



chain of interacting fermions *(quarks, electrons...)*

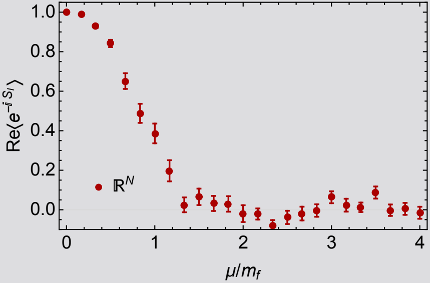
$$S = \int d^2x \bar{\psi}^a \left(\gamma^\mu \partial_\mu + m + \mu\gamma^0\right) \psi^a + \frac{g^2}{2N_f}(\bar{\psi}^a \gamma^\mu \psi^a)(\bar{\psi}^b \gamma_\mu \psi^b)$$

$$\rightarrow \frac{N_F}{2g^2} \int d^2x\, A_\mu A_\mu + \ln\det(\not\partial + \not A + \mu\gamma_0 + m)$$

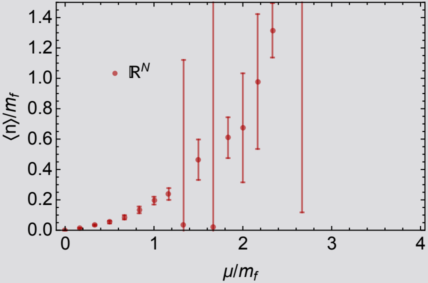
- prototype for dense quark matter: *asymptotically free, sign problem at nonzero density*
- a 2d cousin of the Hubbard model

Many body physics: 2d Thirring model

(GB et. al, Phys. Rev. D95, 014502, 2016)



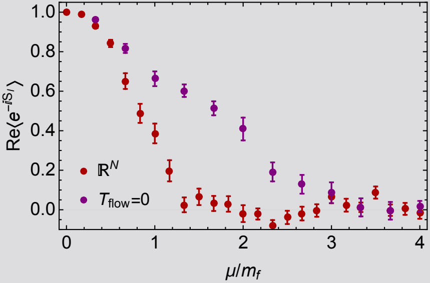
sign problem



equation of state

Many body physics: 2d Thirring model

(GB et. al, Phys. Rev. D95, 014502, 2016)



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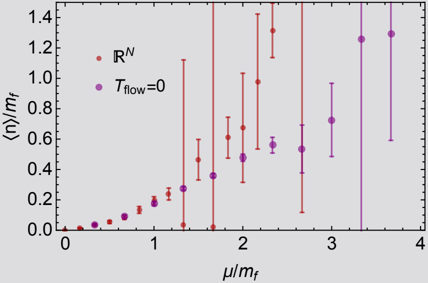

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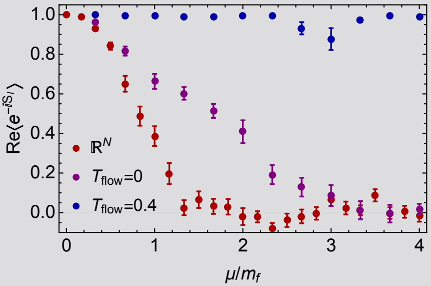

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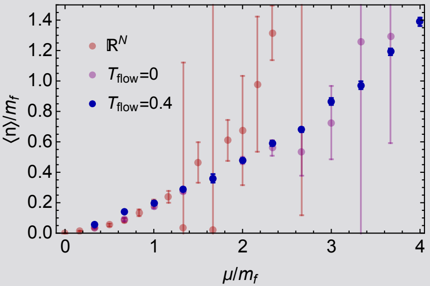


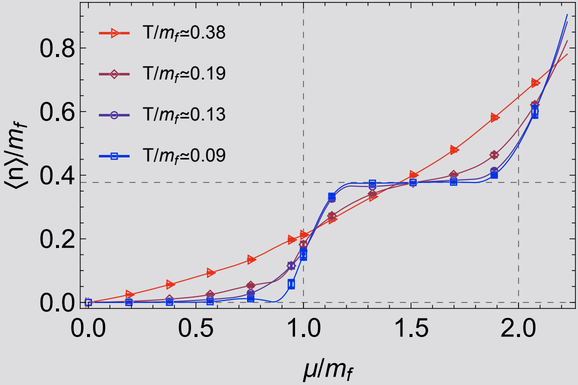
equation of state

Many body physics: 2d Thirring model

(GB et. al, Phys. Rev. D95, 014502, 2016)

equation of state: low temperature limit

particularly bad sign problem: $\langle e^{-iS_I} \rangle_{S_R} \propto e^{-volume/T}$



Many body physics: 2d Thirring model

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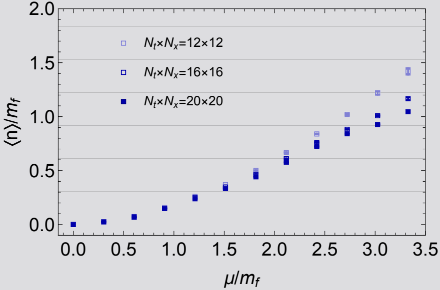
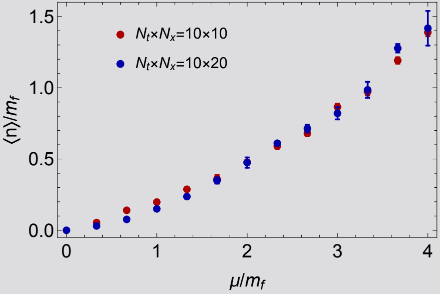
equation of state

continuum limit



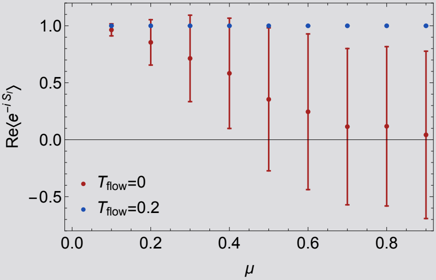
thermodynamic limit



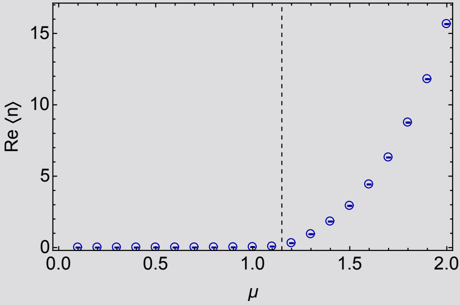
Many body physics: 4d relativistic Bose gas

(GB et. al, Phys. Rev. D94, 045017)

$$\mathcal{L} = |\partial_\mu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu\,(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*) + \lambda|\phi|^4 + h(\phi^1 + \phi^2)$$



sign problem

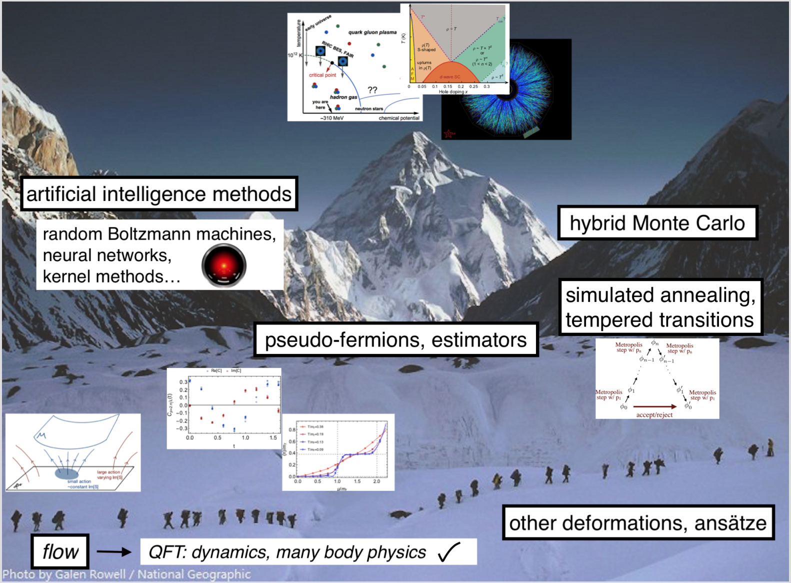


equation of state
at low temperature

Conclusions

- complex space is the natural habitat of QFT
- complex saddles are necessary and contain important physics, most of which have yet to be discovered
- complexification also provides a powerful tool to study dynamical and finite density properties of *strongly interacting systems from first principles*

Future prospects, sign problem



Future prospects, semi-classics

- perturbative - non-perturbative relations in QFT, QM, string theory (quantum geometry, Hecke groups, higher genus...) (building on GB et. al JHEP 1705 (2017) 087)
- poles of the Nekrasov function / conformal block expansion
- implications of Picard-Lefschetz theory for QFT are still not well understood (e.g. supersymmetric Yang-Mills vs complex saddles (see also Ünsal et. al, 2015))

"The shortest path between two truths in the real domain passes through the complex domain."
–Jacques Hadamard

The end

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Exact semi-classics

$$\int_{\mathcal{D}} d\phi \, e^{-\frac{1}{g}S[\phi]} = \int_{\mathcal{J}} d\phi \, e^{-\frac{1}{g}S[\phi]} \quad , \quad \mathcal{J} = \sum_{k \in saddles} n_k \mathcal{J}_k$$

path integral domain = sum over thimbles $\mathcal{J}$

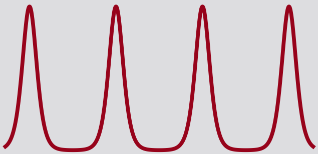
how do we find $\mathcal{J}$?

holomorphic gradient flow:

continuously deform the path integration domain into the combination of Lefschetz thimbles (saddles + fluctuations)

Ghost instantons

another periodic potential



$V(\phi) = \wp(\phi, \mathbb{t})$ "Weierstrass elliptic function"

encodes $\mathcal{N} = 2^*$ SUSY gauge theory in the Nekrasov-Shatashvili limit

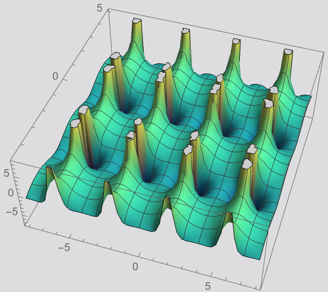
large order growth $\sim n!/(2\tilde{S})^n$ band width $\sim e^{-S/g}$

$\tilde{S} \neq S$, furthermore $\tilde{S} < 0$

what is going on?!?

Ghost instantons

$\wp(\phi, \mathbb{t})$ is also periodic for complex ϕ.



complex instanton (tunneling along imaginary direction)

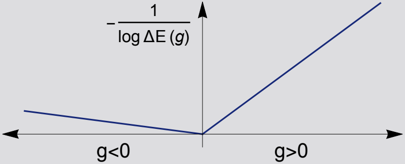
it has action $\tilde{S} < 0 \Rightarrow e^{\frac{|\tilde{S}|}{g}}$ contribution? no.

however, it appears in perturbation expansion.

"ghost instanton" (GB, Dunne, Ünsal)

Ghost instantons and analytic continuation

ghost instantons become physical upon $g \to -g$ and determine the band width



band width changes non-smoothly for $g \to -g$.

physics: quantum phase transition (GB, Dunne, Ünsal)

upshot: complex saddles play a role in analytic continuation of path integrals

similar phenomenon in AdS $\to$ dS? (Hellerman, Maloney, Shenker, et. al)

AGT, NS limit, conformal blocks

(Alday, Gaiotto, Tachikawa; Marshakov et. al.; ...)

$$Z^{inst.}(a;\epsilon_1,\epsilon_2)=\sum_{n=0}^{\infty}\left(\frac{\Lambda^2}{\epsilon_1\epsilon_2}\right)^{2n} Q_\Delta^{-1}([1^n],[1^n]),\qquad Q_\Delta(Y,Y')=\langle\Delta|L_Y\,L_{-Y'}|\Delta\rangle$$

- AGT: $\Delta=\frac{1}{\epsilon_1\epsilon_2}\left(a^2-\frac{(\epsilon_1+\epsilon_2)^2}{4}\right)\quad,\quad c=1-\frac{6(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}$
- $\epsilon_2\to 0$ limit, *twisted superpotential:* (Nekrasov, Shatashvili)

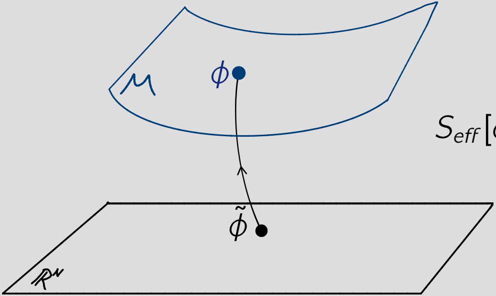
$$\mathcal{W}_{NS}^{inst.}(a;\epsilon_1)\equiv-\frac{\epsilon_1}{4\pi i}\lim_{\epsilon_2\to 0}\epsilon_2\log\left(Z^{inst.}(a,\epsilon_1,\epsilon_2)\right)$$

- identify $\epsilon_1=\hbar=g$, $a=N\hbar/2$

$$u=\frac{i\pi}{2}\Lambda\frac{\partial\mathcal{W}_{NS}^{inst.}}{\partial\Lambda}=\frac{\hbar^2}{8}\left(\frac{8\Lambda^4}{(N^2-1)\,\hbar^4}+\frac{8\Lambda^8\,(5N^2+7)}{(N^2-4)\,(N^2-1)^3\,\hbar^8}+\dots\right)$$

Importance sampling on $\mathcal{M}$

$$\langle \mathcal{O} \rangle = \frac{\int d\phi \, \mathcal{O} e^{-S(\phi)}}{\int d\phi \, e^{-S(\phi)}} = \frac{\int d\tilde{\phi} \, \det\left(\frac{\partial \phi}{\partial \tilde{\phi}}\right) \mathcal{O} \, e^{-S[\phi(\tilde{\phi})]}}{\int d\tilde{\phi} \, \det\left(\frac{\partial \phi}{\partial \tilde{\phi}}\right) \, e^{-S[\phi(\tilde{\phi})]}}$$



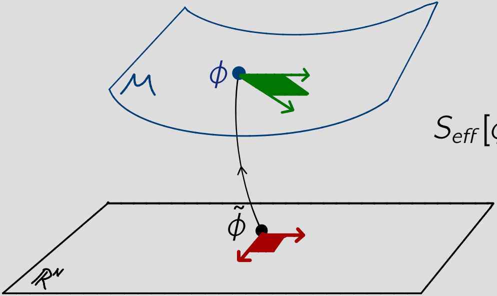
$$S_{eff}[\tilde{\phi}] = \mathbb{R}\mathrm{e}\left(S[\phi(\tilde{\phi})] - \log \det J\right)$$

$$\frac{dJ_{ij}}{d\tau} = \overline{\frac{\partial^2 S}{\partial \phi_i \partial \phi_k} J_{kj}}$$

(GB et. al; PRD 93, 014504, JHEP 05 (2016) 053, PRD 93, 094514)

Importance sampling on $\mathcal{M}$

$$\langle \mathcal{O} \rangle = \frac{\int d\tilde{\phi}\, \mathcal{O} e^{-i\,\mathbb{I}\mathrm{m}(S - \log\det J)} e^{-S_{eff}}}{\int d\tilde{\phi}\, e^{-i\,\mathbb{I}\mathrm{m}(S - \log\det J)}\; e^{-S_{eff}}}$$



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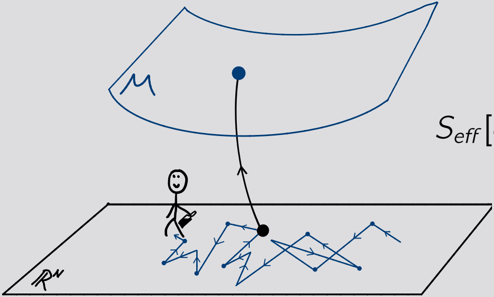
Importance sampling on $\mathcal{M}$

$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O}\, e^{-i\, \mathbb{Im}(S - \log\det J)} \right\rangle_{S_{eff}}}{\left\langle e^{-i\, \mathbb{Im}(S - \log\det J)} \right\rangle_{S_{eff}}}$$



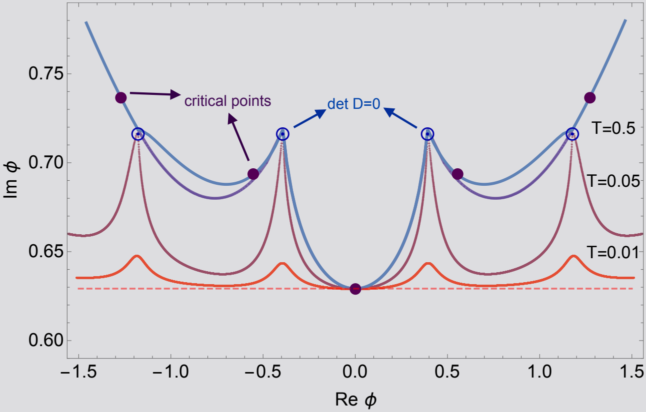
$$S_{eff}[\tilde{\phi}] = \mathbb{Re}\left(S[\phi(\tilde{\phi})] - \log\det J\right)$$

$$\frac{dJ_{ij}}{d\tau} = \overline{\frac{\partial^2 S}{\partial\phi_i \partial\phi_k}} J_{kj}$$

(GB et. al; PRD 93, 014504, JHEP 05 (2016) 053, PRD 93, 094514)

Many body physics: 2d Thirring model

integration manifolds:



projection: $\phi = \frac{1}{L^2} \sum_x A_0(x)$

Many body physics: 4d relativistic Bose gas

complex scalar field: $\phi = \phi^1 + i\phi^2$

$$\mathcal{L} = |\partial_\mu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu \left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*\right) + \lambda |\phi|^4 + h(\phi^1 + \phi^2)$$

discretization:

$$\begin{aligned} S_{lat.} \quad = \quad & \sum_x \left[\left(4 + \frac{m^2}{2}\right) \phi_x^a \phi_x^a + \frac{\lambda}{4} (\phi_x^a \phi_x^a)^2 - h(\phi_x^1 + \phi_x^2) \right. \\ & \left. - \sum_{\nu=1}^{3} \phi_x^a \phi_{x+\hat{\nu}}^a - \cosh \mu \; \phi_x^a \phi_{x+\hat{0}}^a - i \sinh \mu \; \epsilon_{ab} \phi_x^a \phi_{x+\hat{0}}^b \right] \end{aligned}$$

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sign problem here!

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