WKB, Eigenvalue Problems and Quantisation in QM

Gökçe Başar
University of North Carolina, Chapel Hill

Spring school on asymptotic methods and applications
Isaac Newton Institute for Mathematical Sciences
March 22 & 26, 2021

Outline

Lecture I: Basics of WKB and local analysis

Lecture II: Global analysis, Stokes automorphisms, exact quantization

Lecture III: Exact quantization, a geometric approach

References that I follow the closest:

- Delabaere Pham, *Resurgent methods in semi-classical asymptotics*
- Kawai, Takei, *Algebraic Analysis of Singular Perturbation Theory*
- Voros, *Spectre de L'équation de Schrödinger et Méthode BKW*

Exact WKB in a nutshell

Time independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+V(x)\right)\psi(x;\hbar)=E\,\psi(x;\hbar)$$

Wave-functions? , Spectrum ?

$\psi(x,\hbar), E(\hbar)$: Resurgent functions of $\hbar$

$$\hbar^n, e^{-\frac{C}{\hbar}}, \log\hbar\ldots$$

BPS spectra of $\mathcal{N}=2$ SUSY theories [Nekrasov, Shatashvili,…]
wall crossing [Gaiotto, Moore, Nietzke…], cluster algebras [Iwaki, Nakanishi],
Integrable models [Dorey, Dunning, Tatteo…],…

Riccati Equation

$$\left(-\frac{d^2}{dx^2}+\hbar^{-2}Q(x)\right)\psi(x;\hbar)=0, \quad Q(x):=2(V(x)-E)$$

WKB Ansatz: $\psi(x;\hbar):=e^{-\frac{1}{\hbar}\int_{x_0}^{x}P(x;\hbar)dx}$

$$P^2(x;\hbar)-\hbar\frac{dP}{dx}=Q(x)$$

Resurgent expansion: $P(x;\hbar)\sim P_0(x)+\hbar P_1(x)+\hbar^2 P_2(x)+\ldots$

depend on E

*assume *V(x):* polynomial, see [Koike, Schafke] for *V(x)*: rational function

Riccati Equation

$$P(x;\hbar) \sim \sum_{n=0}^{\infty} \hbar^n P_n(x)$$

$$P^2(x;\hbar) - \hbar \frac{dP}{dx} = Q(x)$$

Zeroth order solution: $P_{0,\pm}(x) = \pm\sqrt{Q(x)}$

Two branches ⟷ independent solutions of the Schrödinger eqn.

Once the branch is chosen, the higher order terms are determined recursively (without solving any differential equation!):

$$P_1(x) - \frac{1}{2}\frac{d}{dx}\log P_0(x) = 0$$

$$2P_0(x)P_{n+1}(x) + \frac{dP_n}{dx} + \sum_{k=1}^{n} P_k(x)P_{n+1-k}(x) = 0$$

[Dunham, 1932]

Riccati Equation

We can organize the expansion as

$$P_{\pm}(x;\hbar) := \pm P_{even}(x;\hbar) + P_{odd}(x;\hbar)^*$$

$$\sqrt{Q(x)} + \hbar^2 P_2(x) + \dots \qquad \hbar P_1(x) + \hbar^3 P_3(x) + \dots$$

$$P_{odd}(x) + \frac{\hbar}{2}\frac{d}{dx}\log P_{even}(x) = 0$$

$$\psi(x;\hbar) = \sqrt{\frac{\hbar}{P_{even}(x;\hbar)}}\, e^{\pm\frac{1}{\hbar}\int_{x_0}^{x} P_{even}(x;\hbar)dx} \sim e^{\pm\frac{1}{\hbar}\int_{x_0}^{x}\sqrt{Q(x)}dx} \sum_{n}^{\infty} \psi_{n,\pm}(x)\hbar^{n+1/2}$$

$$P_0(x) = \sqrt{2(V(x)-E)}, \quad P_2(x) = \frac{-5V'(x)^2 + 4(-E+V(x))V''(x)}{32\sqrt{2}(-E+V(x))^{5/2}}, \dots$$

*: the even/odd label here agrees with Delabaere/Pham but is opposite of Takei.

Note: From now on I will drop the "even" subscript from P

WKB and resurgence

$$\psi(x;\hbar) = \sqrt{\frac{\hbar}{P_{even}(x;\hbar)}} e^{\pm\frac{1}{\hbar}\int_{x_0}^{x} P_{even}(x;\hbar)dx} \sim e^{\pm\frac{1}{\hbar}\int_{x_0}^{x}\sqrt{Q(x)}dx} \sum_{n}^{\infty} \psi_{n,\pm}(x)\hbar^{n+1/2}$$

- Each $\psi_{n,\pm}(x)$ is holomorphic near $U = \{x \in \mathbb{C} \,|\, Q(x) \neq 0 \text{ and } Q(x) = \text{holomorphic}\}$
- For any $K \subset U \quad \exists A_K, C_K \quad \text{s.t.} \quad |\psi_{n,\pm}| < A_K C_k^n n!, \quad \forall x \in K$

[Kawai, Takei]

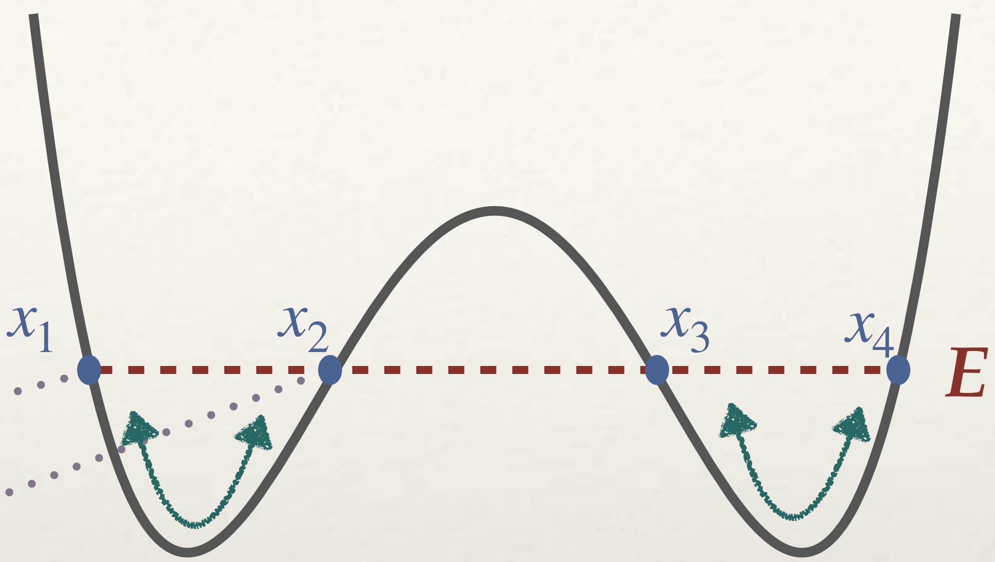
- ψ : resurgent asymptotic series, whose coefficients depend on *x* and *E*.
- It is Borel summable in the absence of Stokes phenomenon
- *Exact WKB : (i)* Patching local WKB expansions in different Stokes regions to construct and analytic function of *x*.

  (ii) exact quantization condition *f*(*E*)=0: determines E as a resurgent function

Classical Mechanics

e.g.: anharmonic oscillator

$$V(x) = \frac{1}{8}(x^2 - 1)^2$$



Turning points

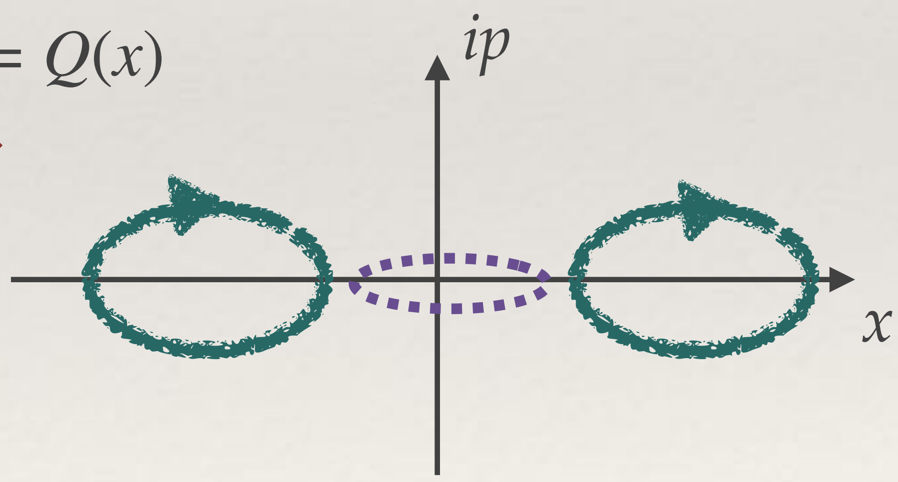
$$Q(x) = 2(V(x) - E) = 0$$

0 th order Riccati: Hamilton-Jacobi *

$$P^2(x;\hbar) - \cancel{\hbar\frac{dP}{dx}} = Q(x)$$

$$-\frac{1}{2}p^2 + V(x) = -\frac{1}{2}\left(\frac{dS_0}{dx}\right)^2 + V(x) = E$$

Classical action $\quad S_0(x) = \int_{x_i}^{x} P_0(x)dx$

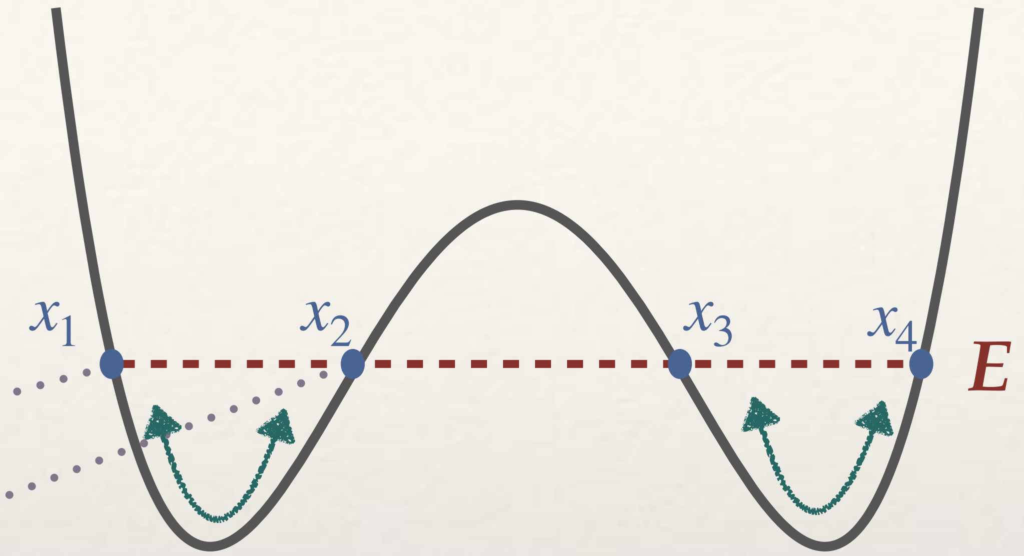


* note that the "momentum", p, differs from the physics convention by a factor of *i*

Classical Mechanics

e.g.: anharmonic oscillator

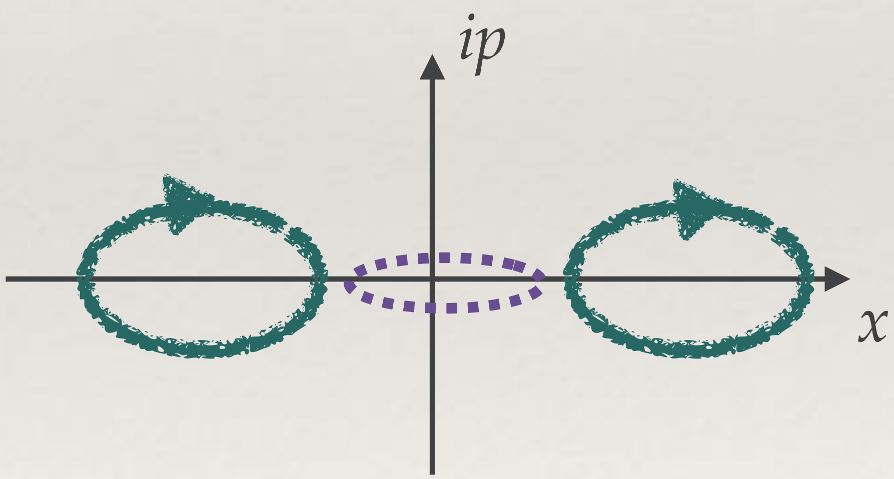
$$V(x) = \frac{1}{8}(x^2-1)^2$$



Turning points

$$Q(x) = 2(V(x) - E) = 0$$

$$S_0(E) = 2\int_{x_1}^{x_2} P_0(x)dx = -2i\int_{x_1}^{x_2}\sqrt{2(E-V(x))}dx$$

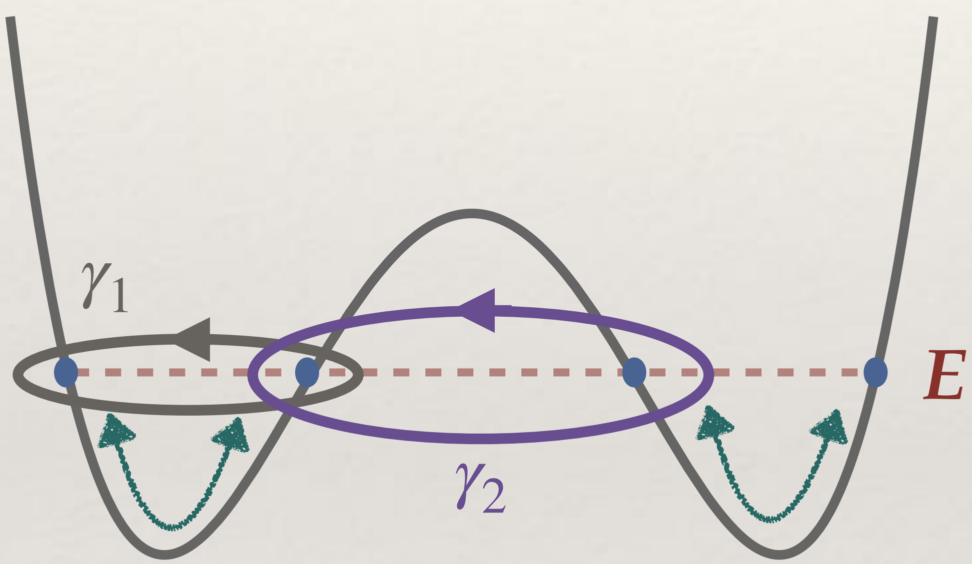


$$\omega_0(E) = \frac{d\mathcal{S}_0(E)}{dE} = -2i\int_{x_1}^{x_2}\frac{1}{\sqrt{2(E-V(x))}}dx = 2\pi i \times \text{(Classical period)}$$

Classical Mechanics

Spectral curve: $\Sigma = \{(p, x) \in \mathbb{C} \,|\, p^2 - Q(x) = 0\}$: *complex hyper-elliptic curve*

e.g.: anharmonic oscillator $\quad p^2 - V(x) - E := p^2 - \prod_{i=1}^{4} (x - x_i(E)) = 0 \quad$ *g=1 elliptic curve*



Moduli space parameter

$$\mathcal{S}_{\gamma_1,0}(E) = \oint_{\gamma_1} P_0(x)dx$$

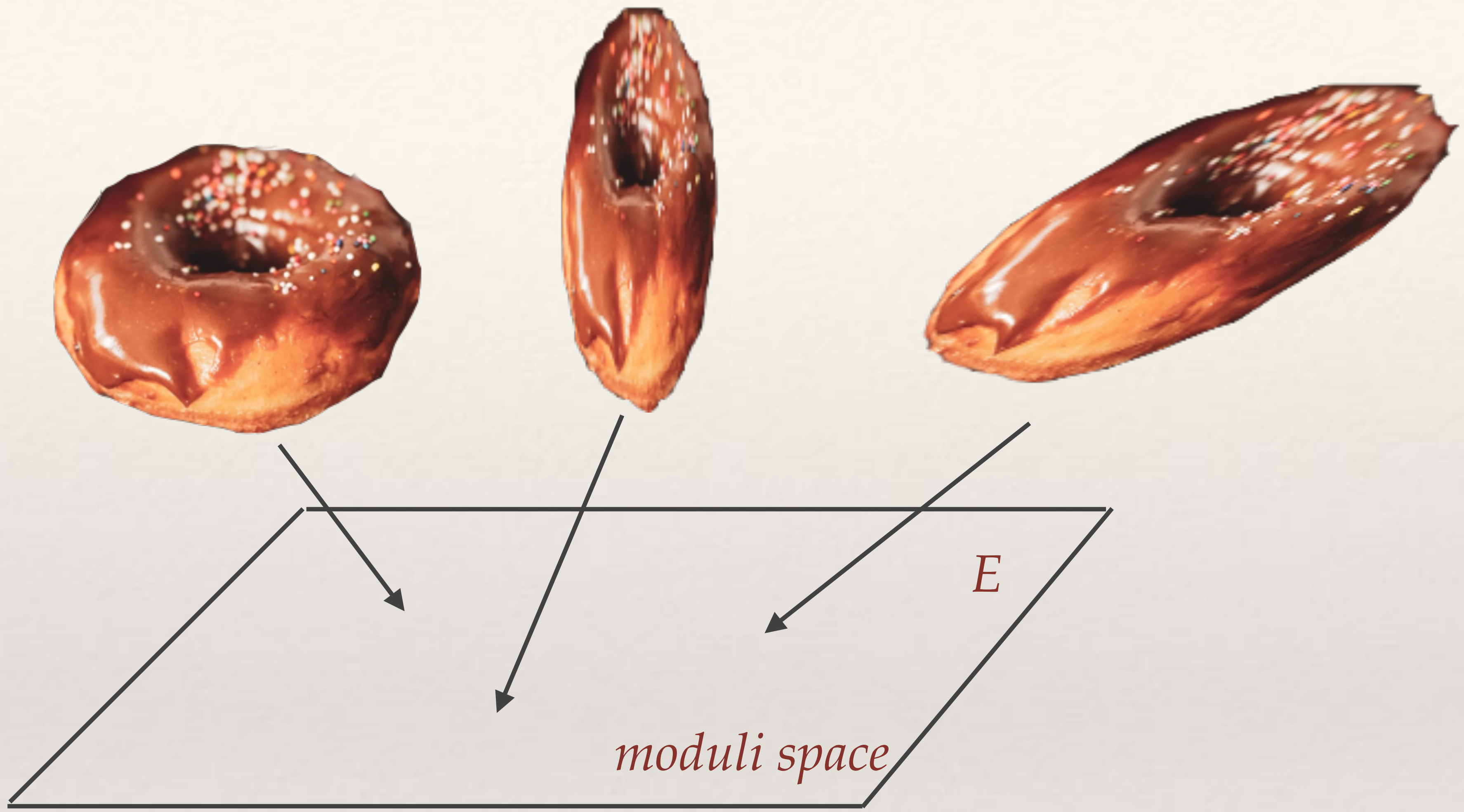
Classical action

$$S_{\gamma_2,0}(E) = \oint_{\gamma_2} P_0(x)dx$$

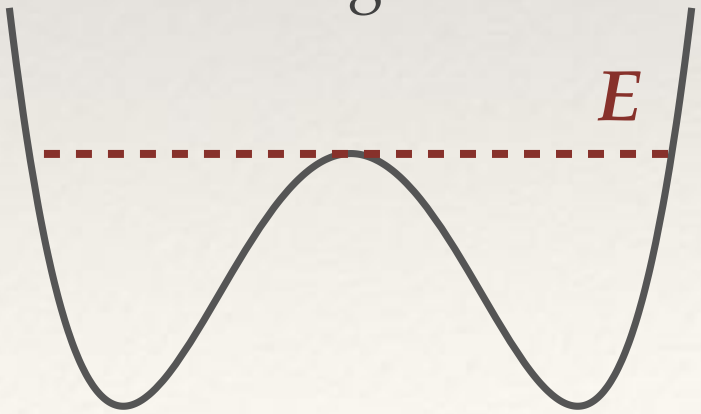
"Tunneling action"

WKB actions $\longleftrightarrow$ $\gamma_i \in H_1(\Sigma)$

Classical Mechanics



e.g.



$$\mathcal{S}_{\gamma_i,0}(E) = \oint_{\gamma_i} P_0(x)dx$$

Solutions of Picard-Fuchs equation

degenerate points

$$\gamma_i = 0$$

Singularities of PF eqn

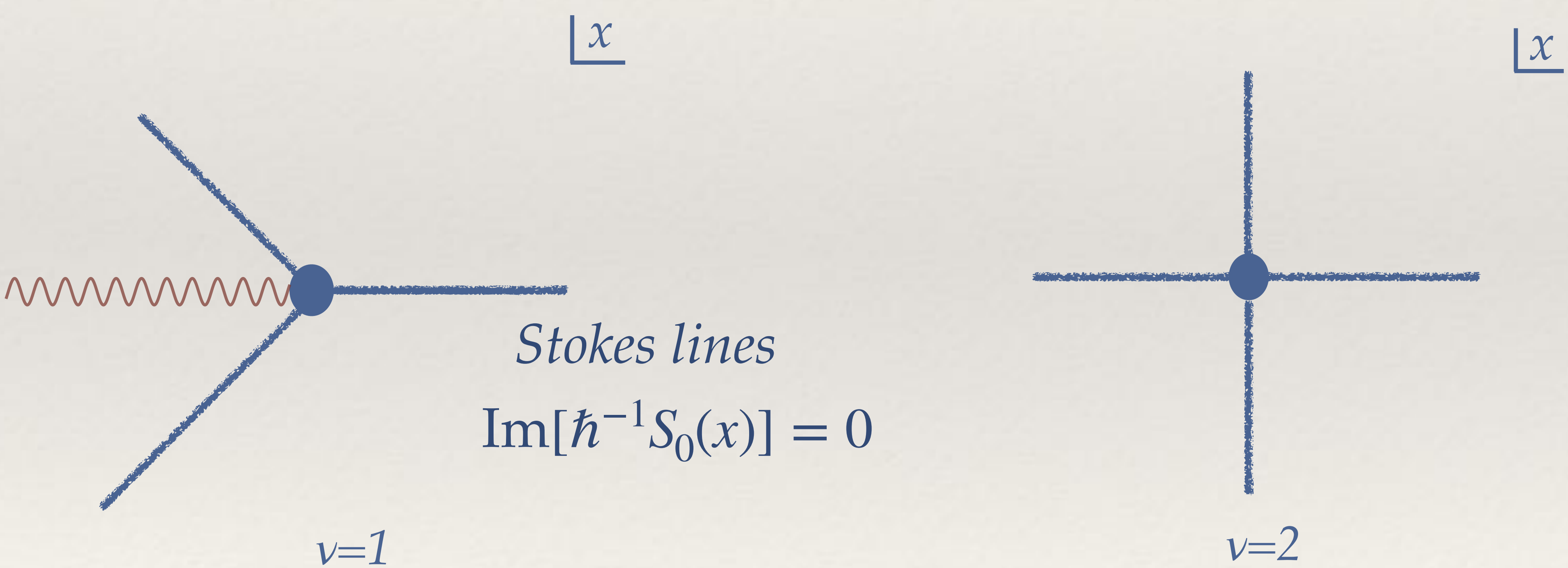
To be continued…

Local analysis: Turning points and Stokes lines

Near a turning point

$$Q(x) \approx (x - x_*)^{\nu}, \quad P_0(x) \approx (x - x_*)^{\nu/2}, \quad S_0(x) \approx (x - x_*)^{\nu/2+1}$$

ν=1: simple turning point, ν=2: double turning point etc…



Steepest descent and Stokes lines

$$x := x_R + i x_I \qquad \frac{1}{\hbar} S_0(x) := f_R(x) + i f_I(x)$$

- Consider the curves, $x(\tau)$, parameterized by τ and satisfy

$$\frac{dx}{d\tau} = \overline{\frac{\partial f(x(\tau))}{\partial x}}$$

$$\Rightarrow \frac{df_R}{d\tau} = \left| \frac{\partial f}{\partial x} \right|^2 > 0, \quad \frac{df_I}{d\tau} = 0$$

Steepest descent curves

- $\mathrm{Re}\left[\frac{1}{\hbar} S_0(x)\right]$ increases fastest
- $e^{-\frac{1}{\hbar} S_0(x)}$ decreases fastest

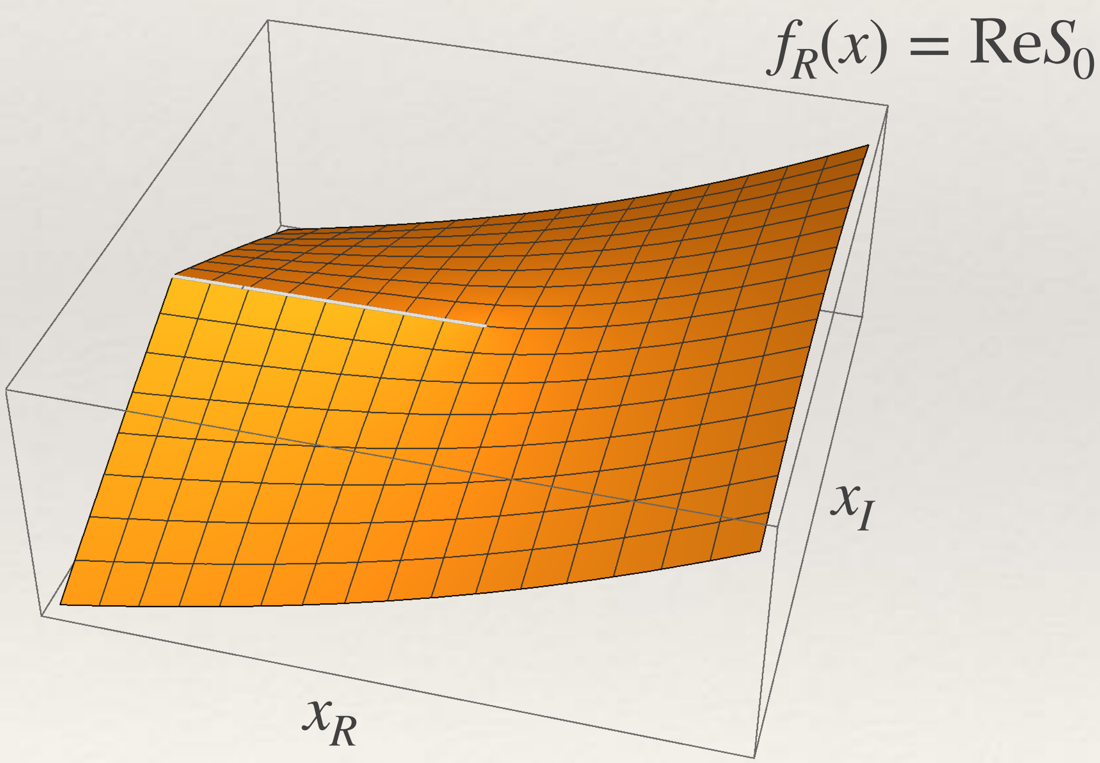
"steepest ascent","Lefschetz thimbles",
"fading lines" [DDP] , ….

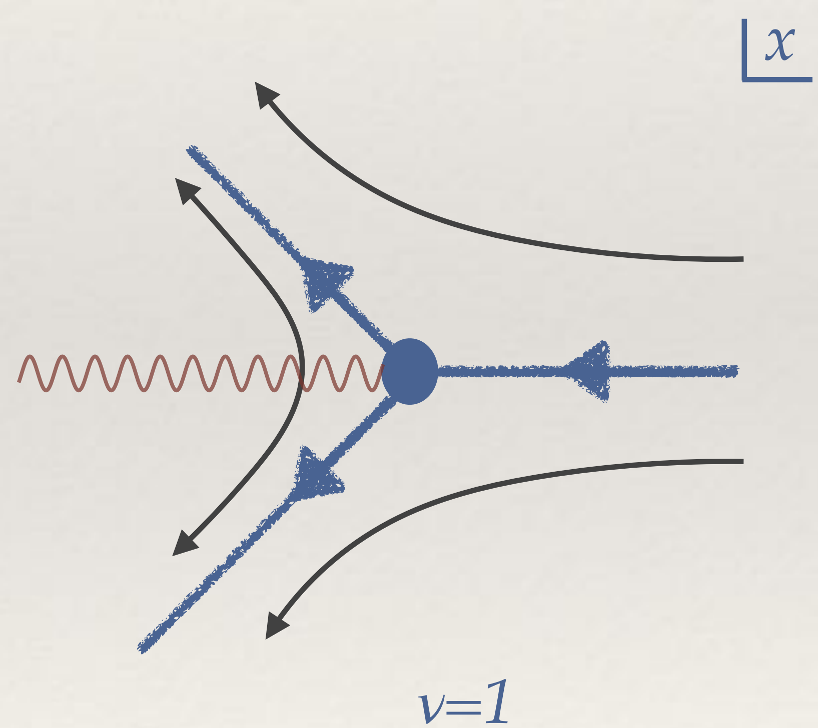
exercise: show that $x(\tau)$ satisfy a gradient flow equation for $S_{0,R}$ and a Hamiltonian equation where $S_{0,I}$ is the Hamiltonian and (x_R, x_I) is the canonical pair

Steepest descent and Stokes lines

recall $\frac{\partial S_0}{\partial x} = P_0(x)$ Turning points: *fixed points* of flow

- Around a simple turning point $P_0(x) = \sqrt{x}, \quad S_0(x) = 2/3x^{3/2}$

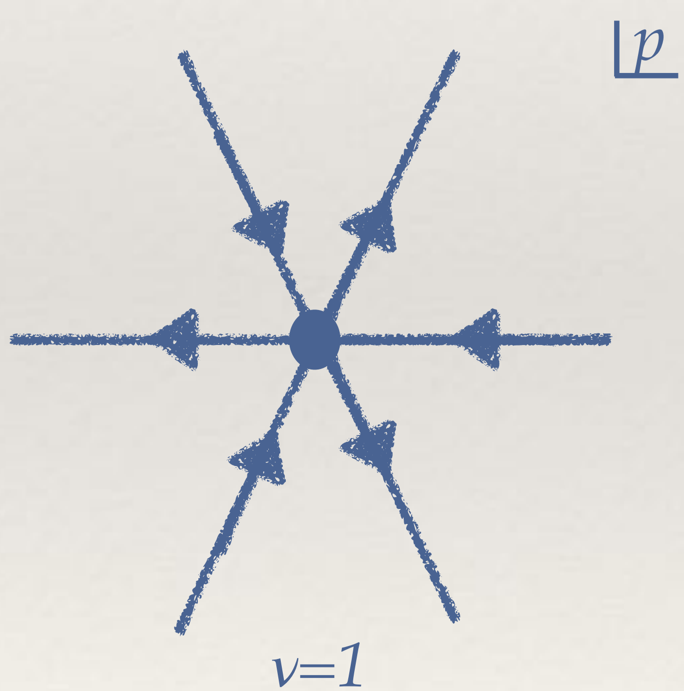


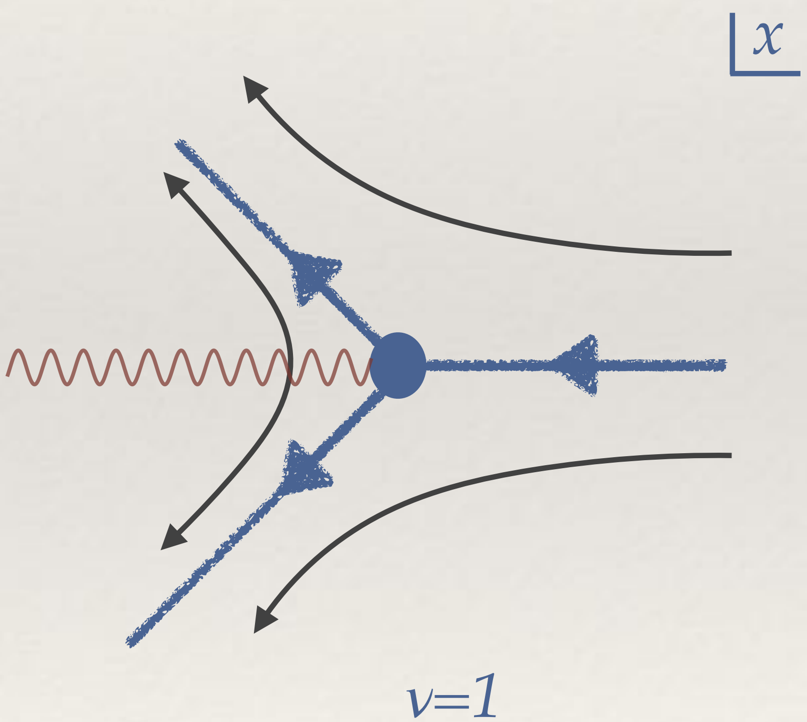


Steepest descent and Stokes lines

recall $\dfrac{\partial S_0}{\partial x} = P_0(x)$ Turning points: *fixed points* of flow

- Around a simple turning point $P_0(x) = \sqrt{x}, \quad S_0(x) = 2/3x^{3/2}$

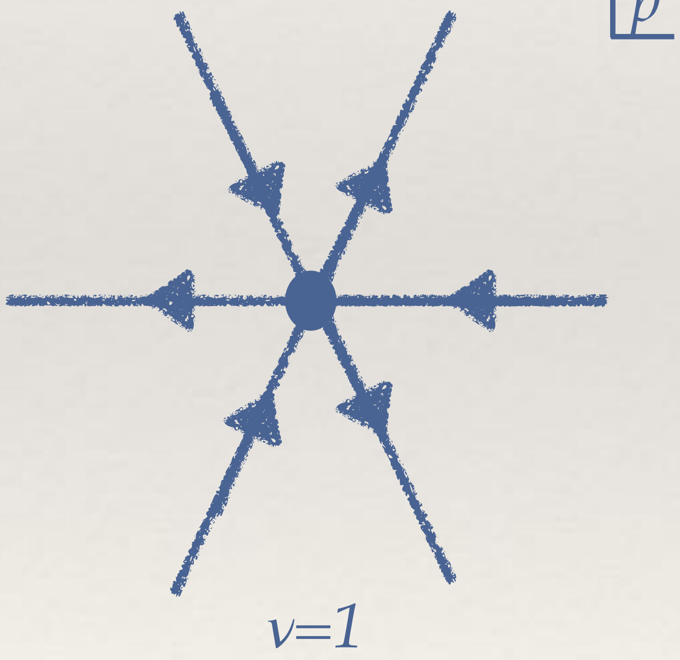


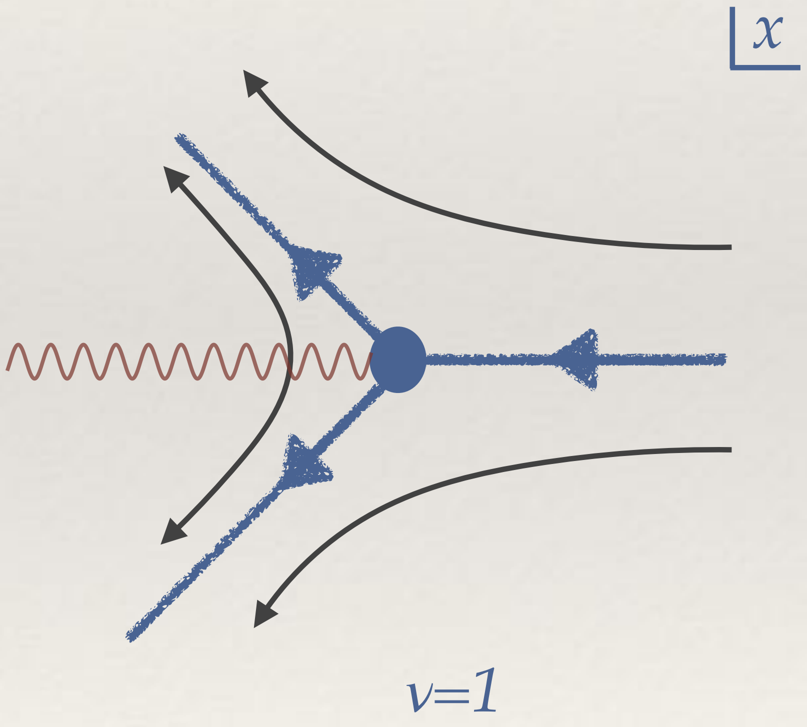


Steepest descent and Stokes lines

recall $\dfrac{\partial S_0}{\partial x} = P_0(x)$ Turning points: *fixed points* of flow

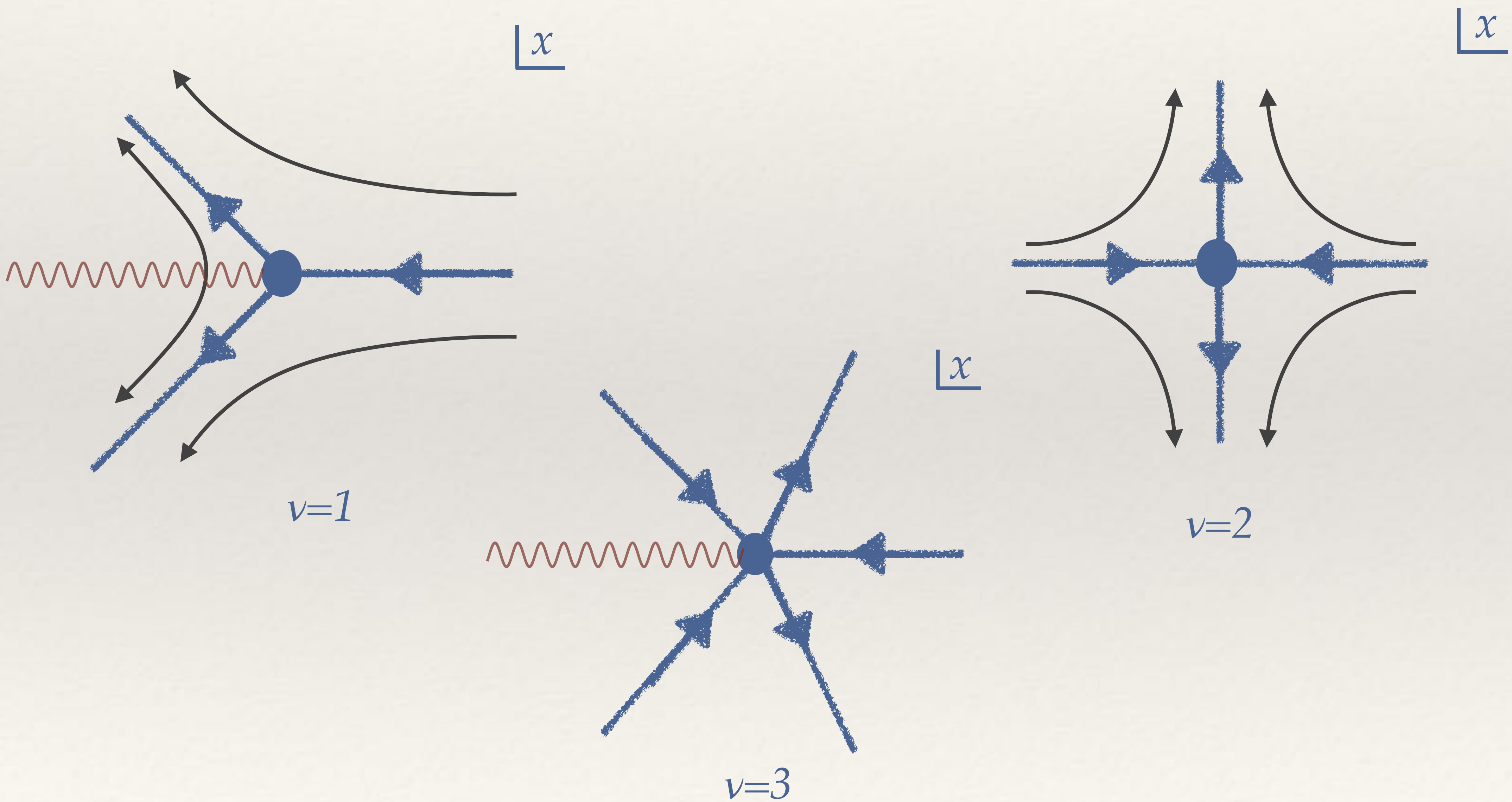
- Around a simple turning point $P_0(x) = \sqrt{x}, \quad S_0(x) = 2/3x^{3/2}$





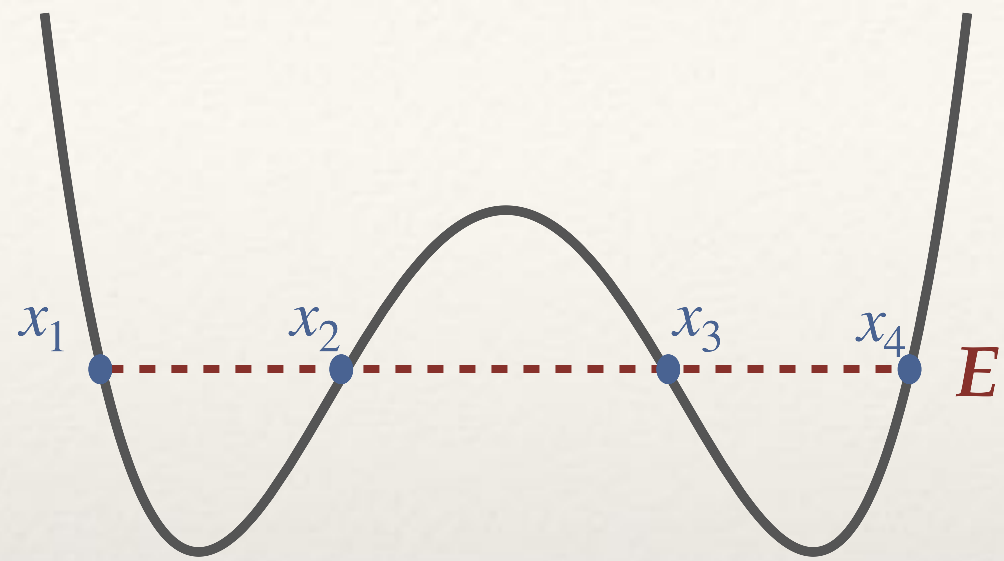
Steepest descent and Stokes lines

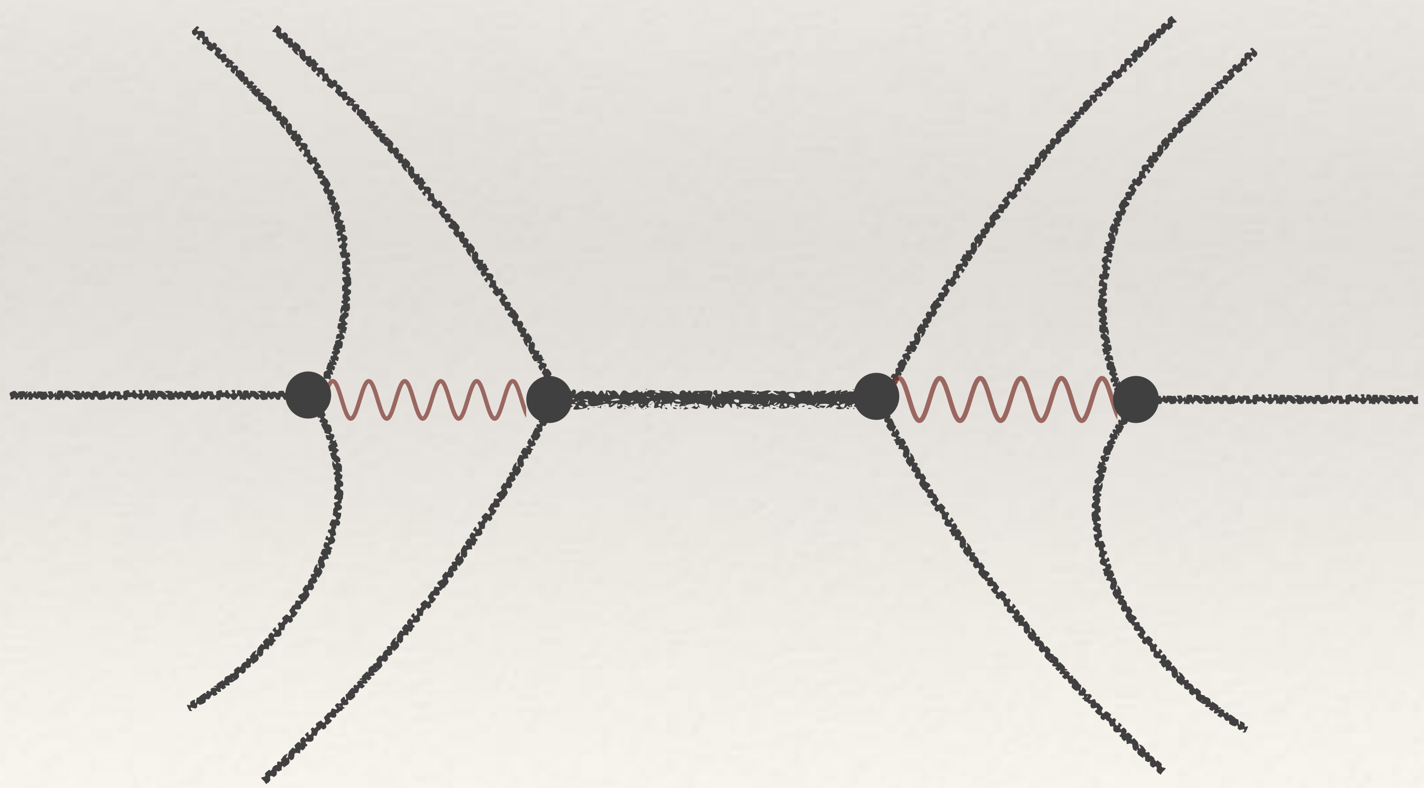
Stokes line $\mathrm{Im}[\hbar^{-1} S_0(x)] = 0$ $\Leftrightarrow$ Steepest descent curves emanating from a turning pt.



Steepest descent and Stokes lines

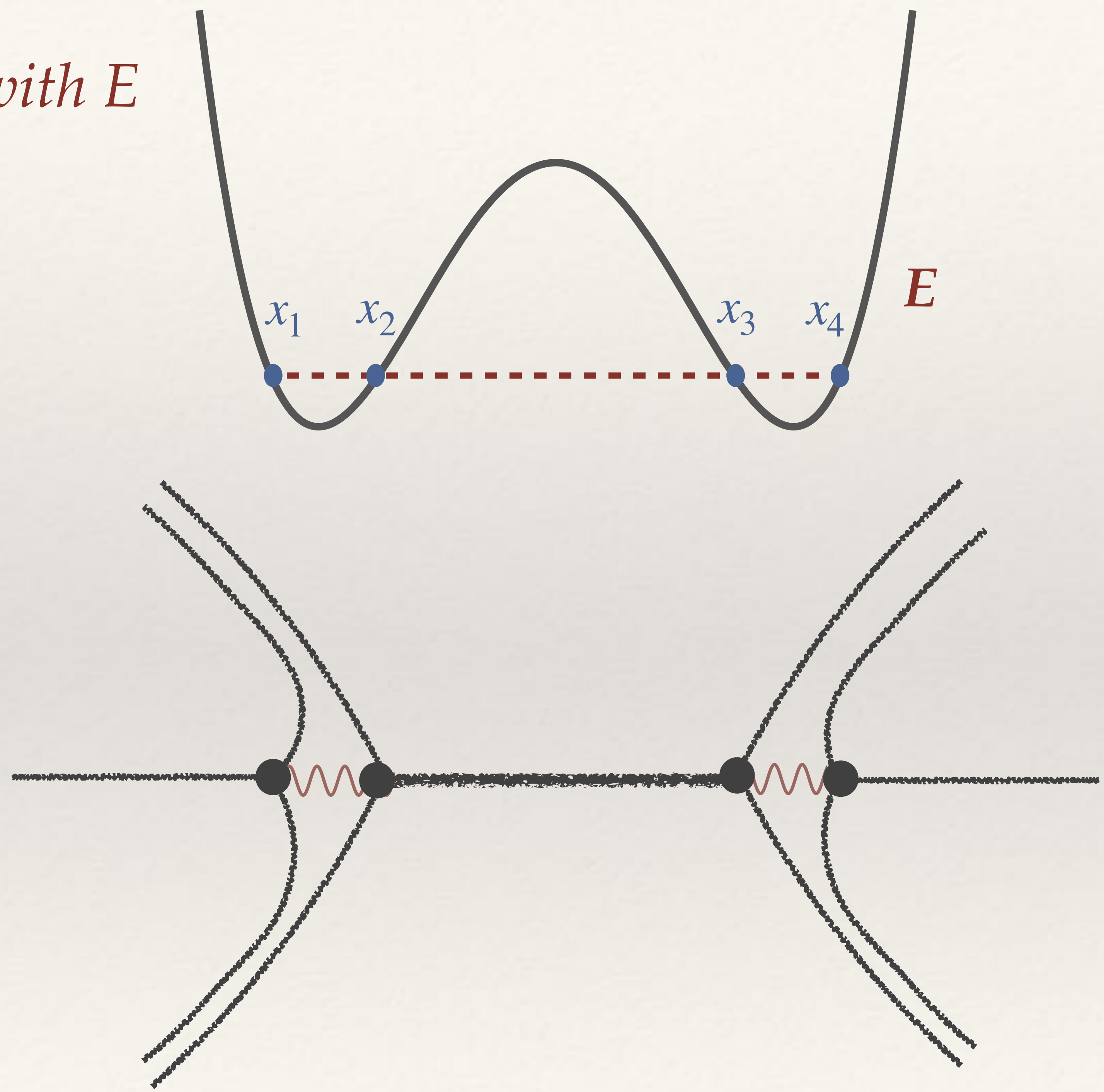
Stokes lines move with E





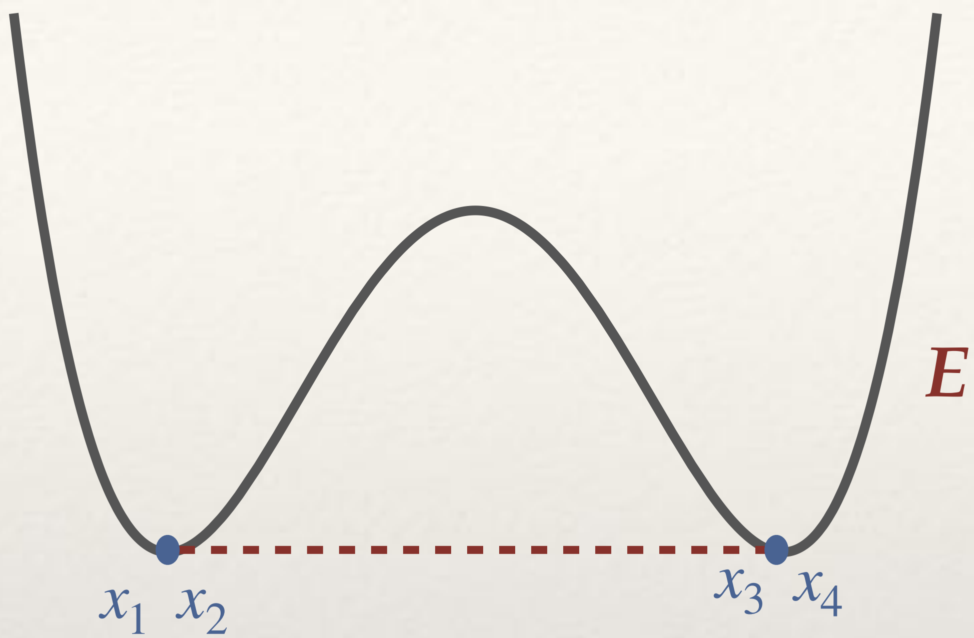
Steepest descent and Stokes lines

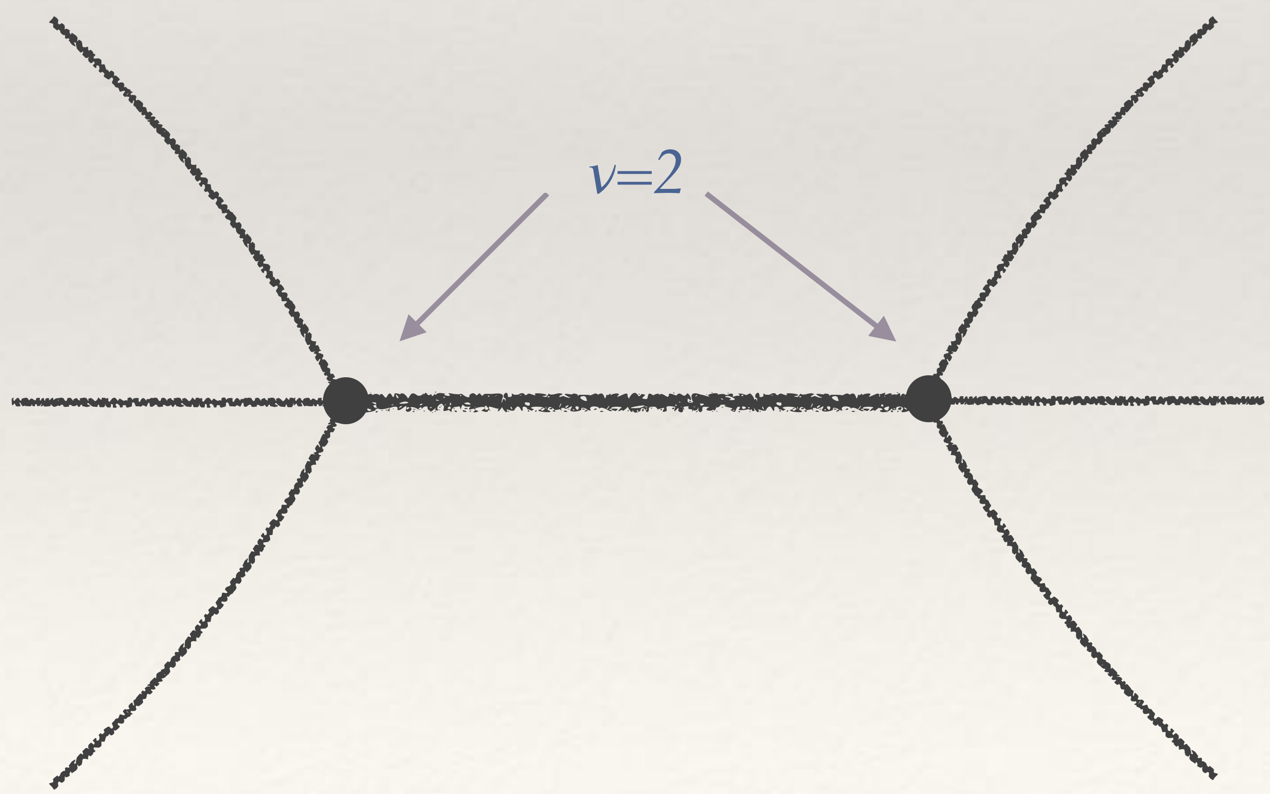
Stokes lines move with E



Steepest descent and Stokes lines

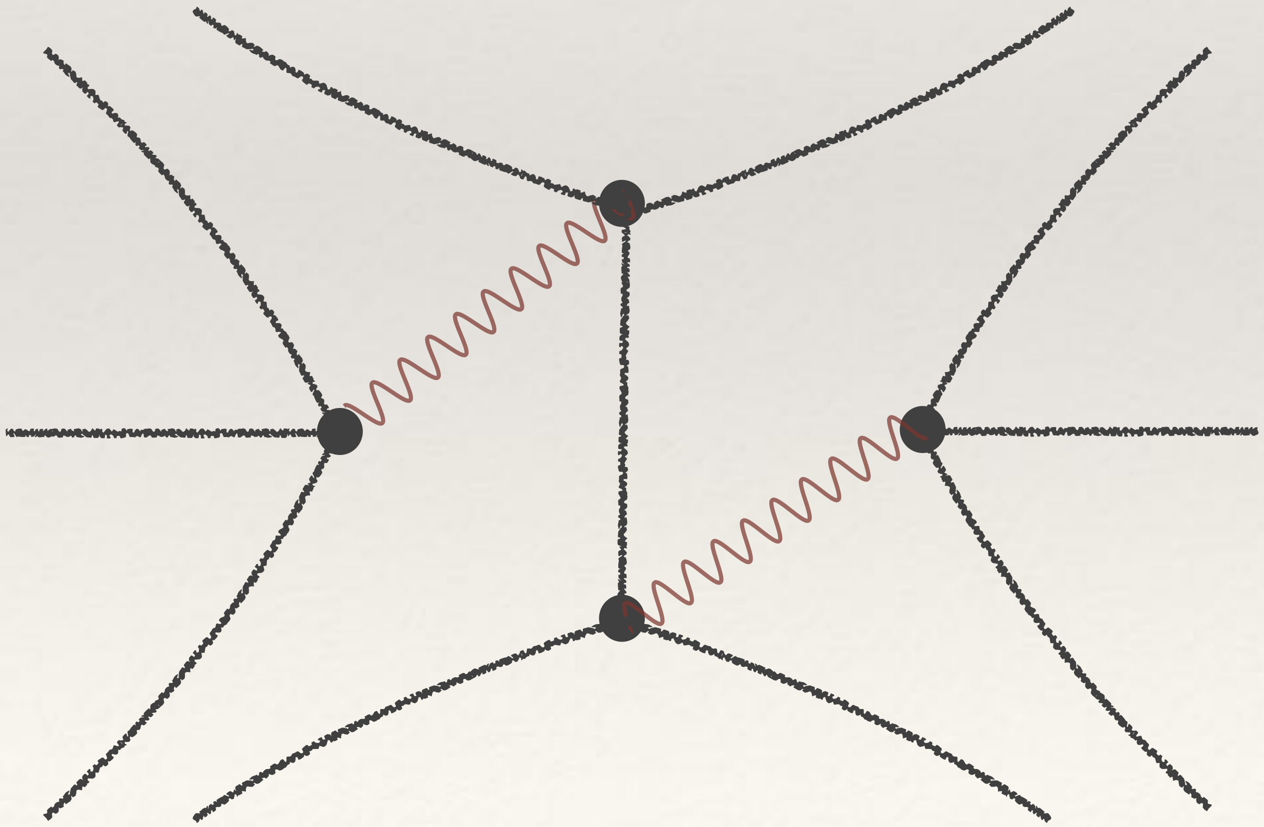
Stokes lines move with E





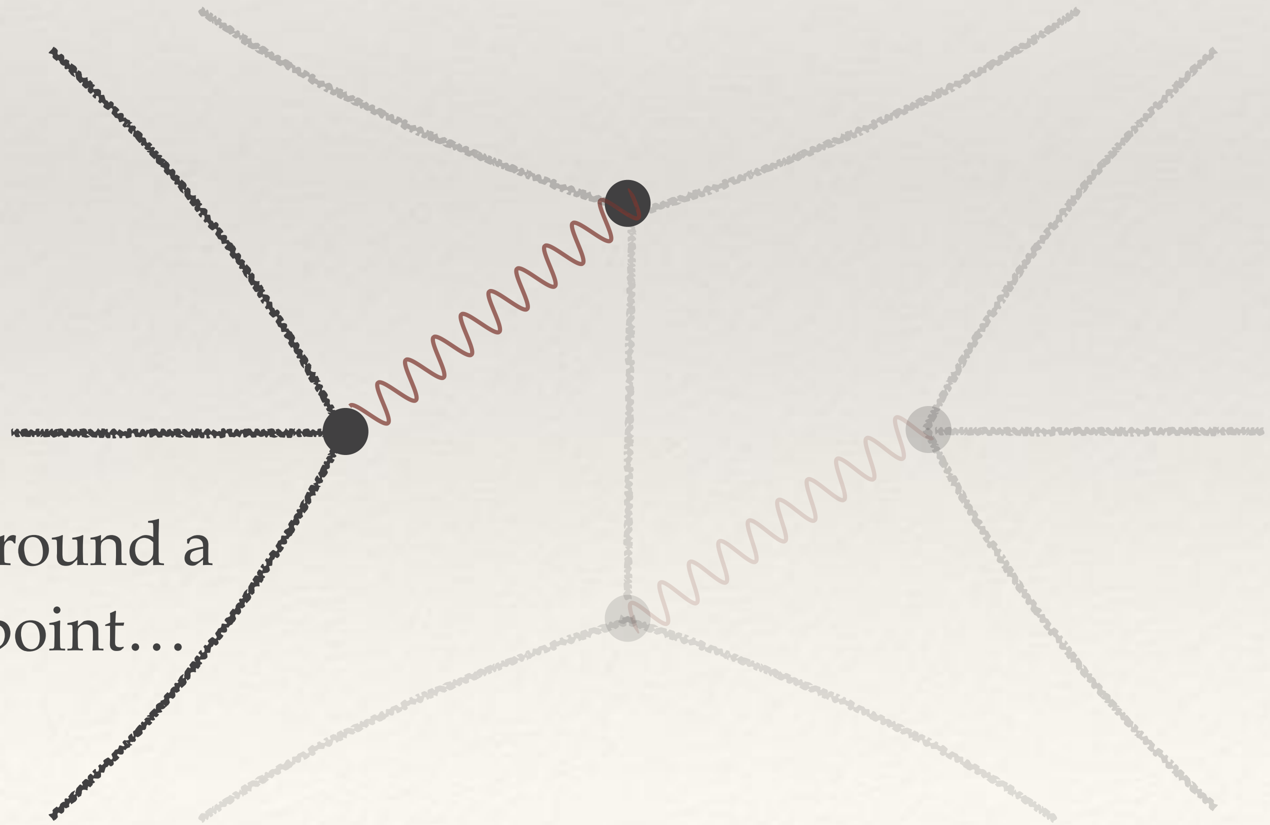
Steepest descent and Stokes lines

Stokes lines move with E



Steepest descent and Stokes lines

Stokes lines move with E



Let's see what happens around a
generic simple turning point…

Borel summation

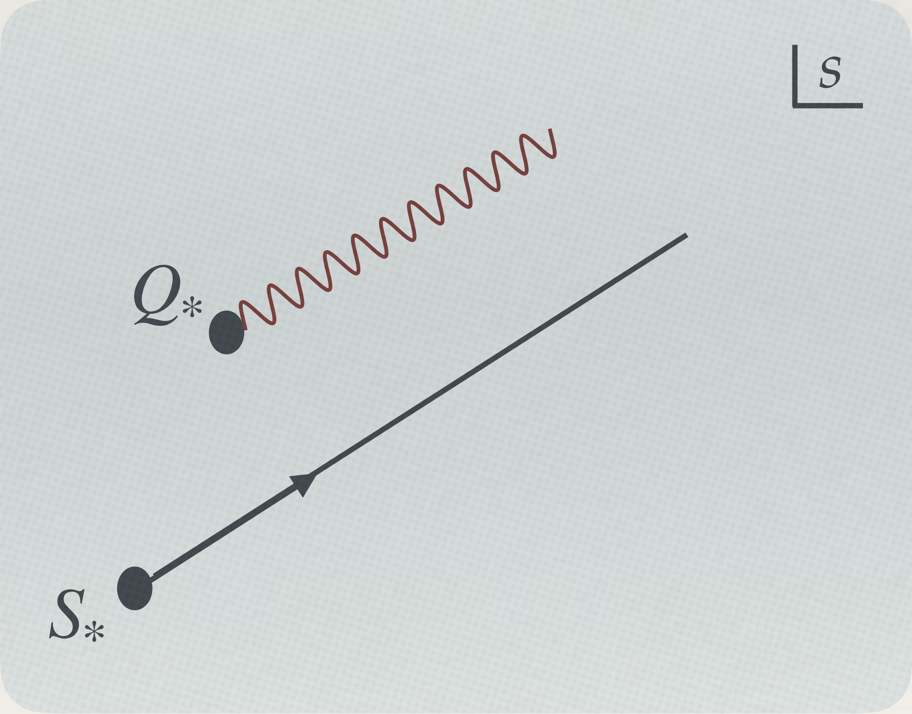
$$f(\hbar) \sim e^{-\frac{S_*}{\hbar}} \sum_{n=0}^{\infty} c_n \hbar^n$$

asymptotic series

$$\mathcal{B}f(s) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n)} (s - S_*)^{n-1}$$

$$\mathcal{S}_\theta[\mathcal{B}f](\hbar) = \int_{S_*}^{e^{i\theta}\infty} ds\, e^{-\frac{s}{\hbar}} \mathcal{B}f(s) \qquad \theta := \arg \hbar$$

- f is *Borel summable* if there are no singularities along the integration contour



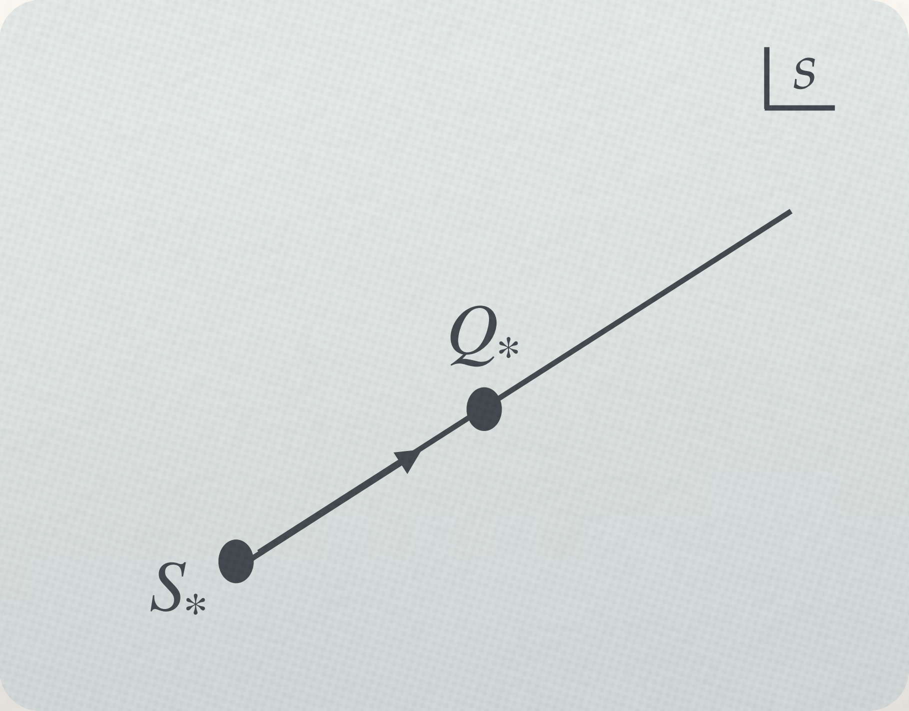
Note: from now on I will simply use $\mathcal{S}_\theta \psi$ (or $\mathcal{S}\psi$) to denote Borel summation

Borel summation

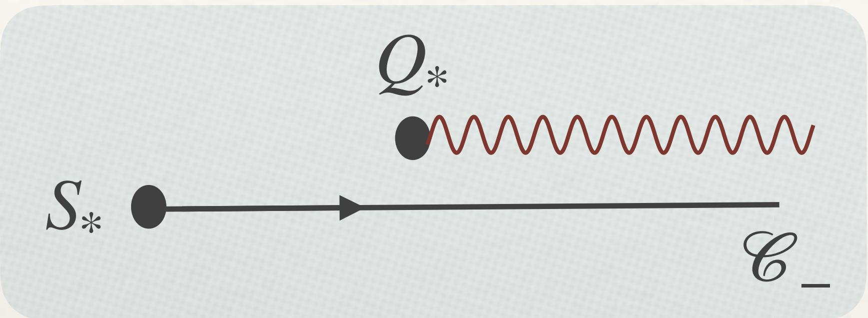
- f is not *Borel summable* if there are singularities along the integration contour

- This might happen for certain values of θ or when the location of the singularities S_*, Q_* depend on some other parameters in the problem (*moving singularities*) . In the WKB problem both of these things happen.

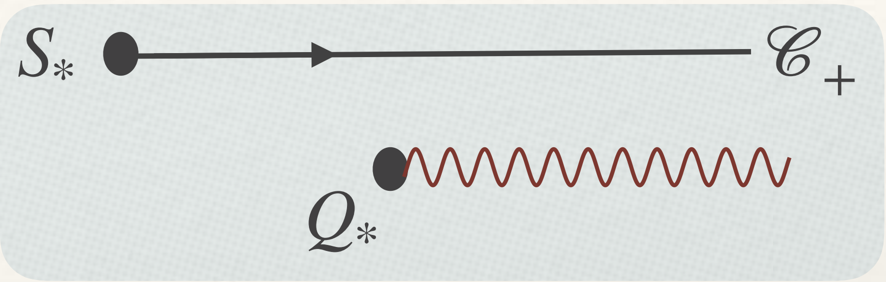
- We can slightly change these parameters to move the singularity out of the way: *Lateral Borel summation*



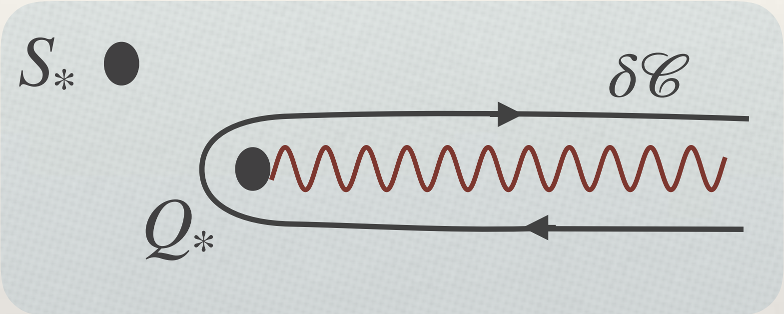
Lateral Borel summations



$$\mathcal{S}_-f(\hbar) := \int_{\mathscr{C}_-} ds\, e^{-\frac{s}{\hbar}}\, \mathscr{B}f(s)$$



$$\mathcal{S}_+f(\hbar) := \int_{\mathscr{C}_+} ds\, e^{-\frac{s}{\hbar}}\, \mathscr{B}f(s)$$



Stokes phenomenon:

$$\mathcal{S}_+f(\hbar) - \mathcal{S}_-f(\hbar) = \int_{\delta\mathscr{C}} ds\, e^{-\frac{s}{\hbar}}\, \mathscr{B}f(s) := ie^{-\frac{1}{\hbar}Q_*}\mathcal{S}_-\, f_Q(\hbar)$$

exponentially suppressed *resurgent function*

Alien derivative: $\Delta_{Q_*}f = if_Q$

Stokes automorphism: $\mathfrak{S} = \mathcal{S}_+ \circ \mathcal{S}_-^{-} = e^{\dot{\Delta}_{Q_*}}$

pointed alien derivative: $\dot{\Delta}_{Q_*} := e^{-\frac{1}{\hbar}Q_*}\Delta_{Q_*}$

generalize to multiple singularities
see e.g. [Aniceto, Basar, Schiappa,
A Primer on Resurgent Transseries and Their Asymptotics]

Borel summation for WKB

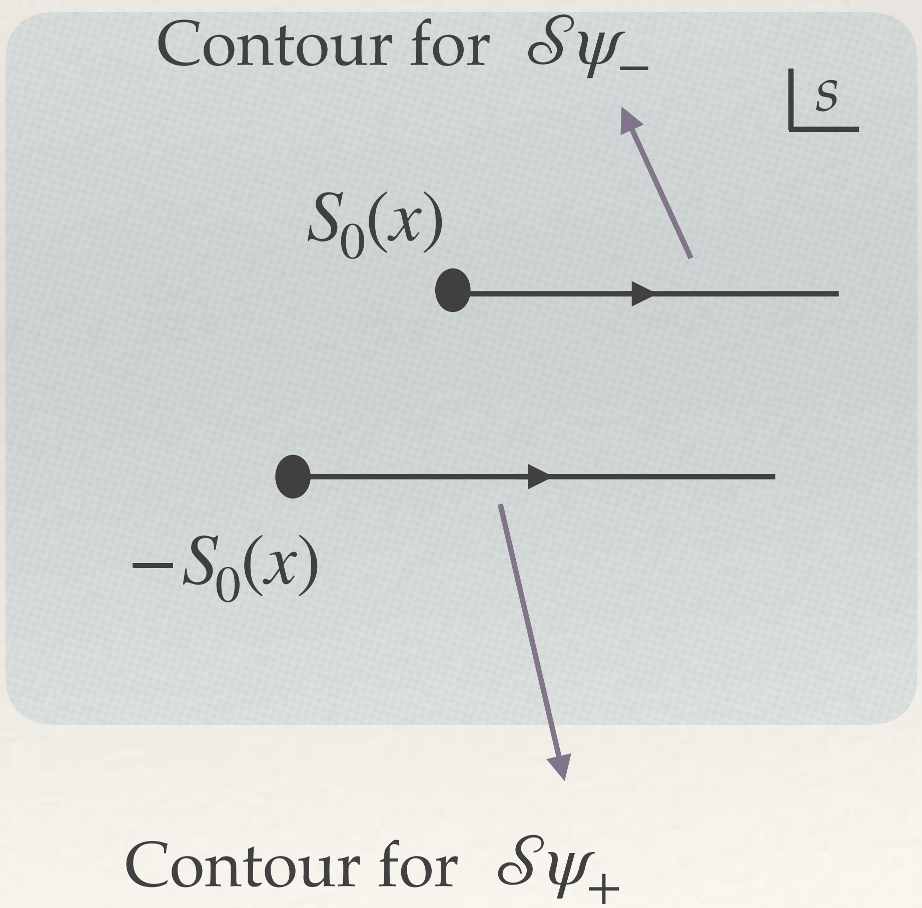
$$\psi(x,\hbar) = c_+\psi_+(x;\hbar) + c_-\psi_-(x;\hbar)$$

$$\psi_\pm(x;\hbar) \sim e^{\pm\frac{1}{\hbar}S_0(x)} \sum_n^\infty \psi_{n,\pm}(x)\hbar^{n+1/2} \qquad \mathcal{S}_\theta[\mathcal{B}\psi_\pm](\hbar) = \int_{\pm S_0(x)}^{e^{i\theta}\infty} ds\, e^{-\frac{s}{\hbar}}\, \mathcal{B}[\psi(x)](s)$$

- Moving singularities: positions depend on *x, E*
- Let's assume *E* is generic (all turning points are simple)

$\psi(x;\hbar)$ is Borel summable as long as

$$\mathrm{Im}[\hbar^{-1}S_0(x)] \neq 0$$

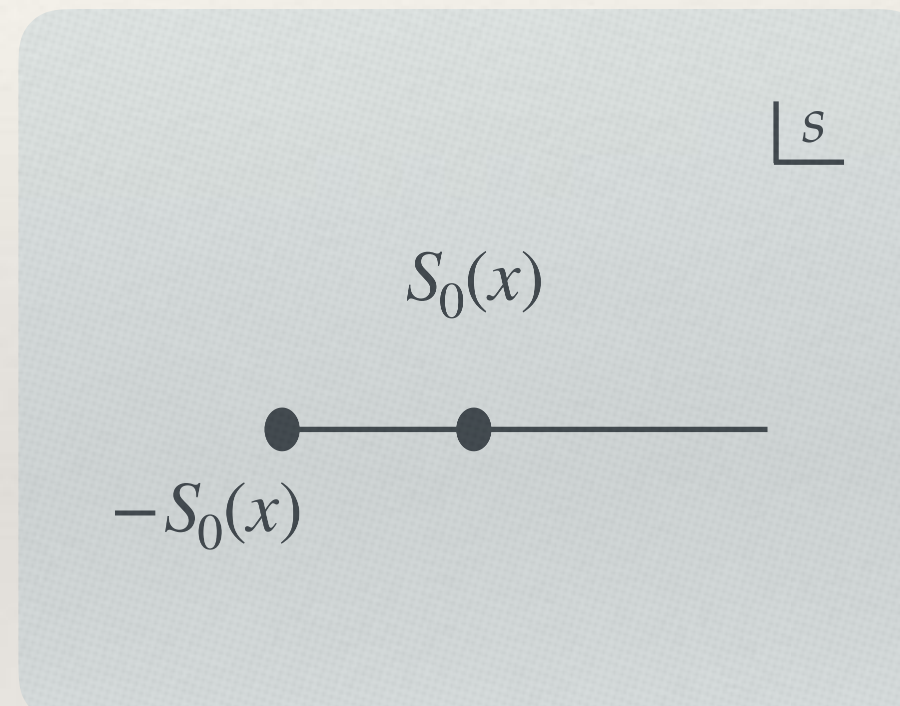


Borel summation

$$\psi_\pm(x;\hbar) \sim e^{\pm\frac{1}{\hbar}S_0(x)} \sum_n^\infty \psi_{n,\pm}(x)\hbar^{n+1/2}$$

$$\mathcal{S}_\theta \psi_\pm(\hbar) = \int_{\pm S_0(x)}^{e^{i\theta}\infty} ds\, e^{-\frac{s}{\hbar}} \mathcal{B}[\psi(x)](s)$$

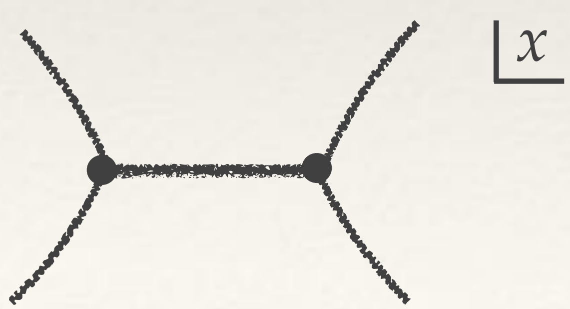
- Moving singularities: positions depend on x, E
- Let's assume E is generic (all turning points are simple)



$\psi(x;\hbar)$ is not Borel summable when

$$\mathrm{Im}[\hbar^{-1}S_0(x)] = 0$$

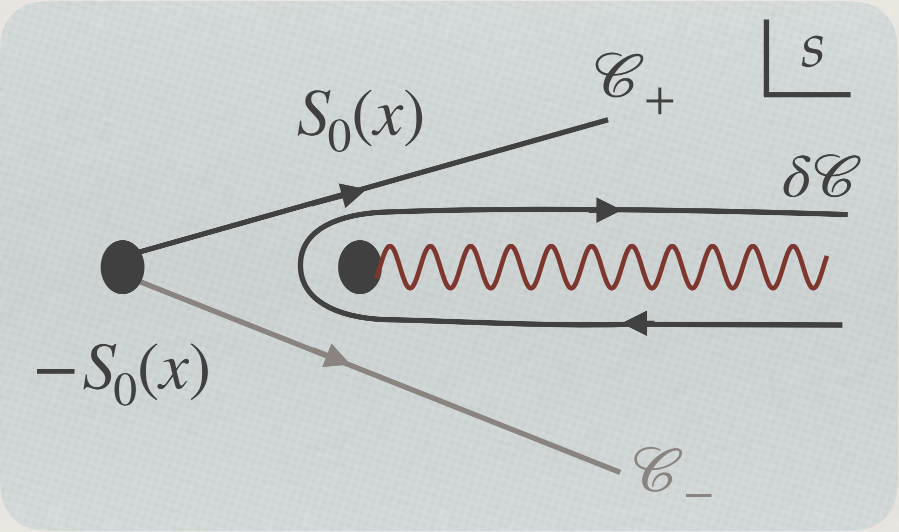
- This happens when two turning points are by a Stokes line ("*degenerate Stokes line*")



Borel plane: crossing the Stokes line

$$\psi_\pm(x;\hbar) \sim e^{\pm\frac{1}{\hbar}S_0(x)} \sum_n^\infty \psi_{n,\pm}(x)\hbar^{n+1/2} \qquad \mathcal{S}_\theta \psi_\pm(\hbar) = \int_{\pm S_0(x)}^{e^{i\theta}\infty} ds\, e^{-\frac{s}{\hbar}} \mathscr{B}[\psi(x)](s)$$

- Assume $\theta = 0, \quad \hbar > 0, \quad \mathrm{Re} S_0(x) > 0 \to \psi_+$: exp. large, $\quad \psi_-$: exp. small



$$\psi_+ \to \psi_+ + i\psi_-$$

$$\psi_- \to \psi_-$$

$$\Delta_{S_0(x)}\psi_+ = i\psi_-$$

$$\Delta_s \psi_- = 0$$

alternatively

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

Strategy: 1) Analyze the Stokes phenomena around each turning point to construct locally
2) Patch the local solutions to construct the global wave-function.

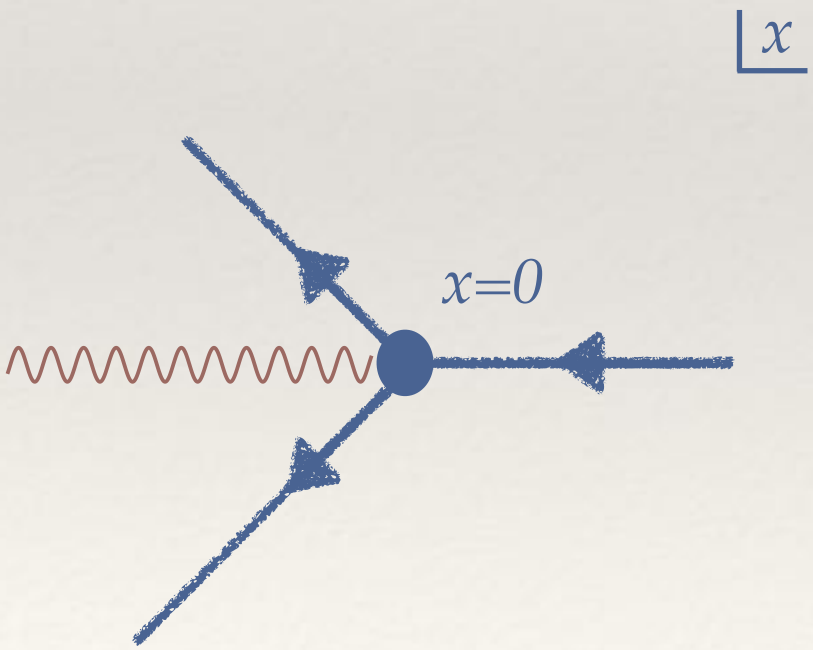
Stokes automorphisms, local analysis

- Let's analyze the Stokes phenomena near a turning point

$$2(V(x) - E) \approx c(x - x_*)$$

- shift, rescale x such that the turning point is at $x=0$

$$\left(-\hbar^2 \frac{d^2}{dx^2} + x\right)\psi(x) = 0 \qquad \textit{Airy equation}$$



classical action: $S_0(x) = \int_0^x \sqrt{x'}dx' = \frac{2}{3}x^{3/2}$

Resurgent expansion: $\psi_\pm(x) = e^{\pm\hbar\frac{2}{3}x^{3/2}} \sum_{n=0}^{\infty} \psi_{n\pm}(x)\hbar^{n+\frac{1}{2}}$

$$\psi = c_+\psi_+ + c_-\psi_-$$

Airy equation

From Riccati recursion relations, $P_{n+1}(x) = \dfrac{1}{2P_0(x)}\left(\dfrac{dP_n}{dx} - \sum_{k=1}^{n} P_k(x)P_{n+1-k}(x)\right)$

$$P_0 = \sqrt{x}, P_1 = (2x)^{-1}, \quad P_n(x) \propto x^{-1-3/2(n-1)}, \psi_n(x) \propto x^{-1/4-3/2n}$$

Exercise: find the coefficients

Borel transform

$$\mathscr{B}\psi_\pm(s) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{c_{n\pm}}{\Gamma(n+1/2)}\left(\frac{s}{x^{3/2}} \pm \frac{2}{3}\right)^{n-1/2} := \frac{1}{x}B_\pm(sx^{-3/2})$$

Airy equation

mostly from [Kawai,Takei]

$$\left(-\hbar^2\frac{d^2}{dx^2}+x\right)\psi(x)=0 \quad \xrightarrow{\text{Borel tr.}} \quad \left(\frac{\partial^2}{\partial x^2}-x\frac{\partial^2}{\partial s^2}\right)\frac{1}{x}B_\pm(sx^{-3/2})=0$$

$$8B(\hat{s})+27\hat{s}\frac{dB}{d\hat{s}}+(9\hat{s}^2-4)\frac{d^2B}{d\hat{s}^2}=0 \qquad \hat{s}:=sx^{3/2}$$

Hypergeometric differential equation

$B_\pm(s)$: independent solutions

$$\mathscr{B}\psi_\pm(s)\propto\frac{1}{x}\left(\frac{3s}{4x^{3/2}}\pm\frac{1}{2}\right)^{-1/2}\ {}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};\frac{1}{2}\pm\frac{3s}{4x^{3/2}}\right)$$

Exercise: derive this from the explicit coefficients

Airy equation

$$\mathscr{B}\psi_+(s) \propto \frac{\sqrt{u}}{x}\,{}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};u\right),\quad \mathscr{B}\psi_-(s) \propto \frac{\sqrt{u-1}}{x}\,{}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};1-u\right)\qquad u := \frac{1}{2}+\frac{3s}{4x^{3/2}}$$

dlmf.nist.gov


Chapter 15 Hypergeometric Function

A. B. Olde Daalhuis

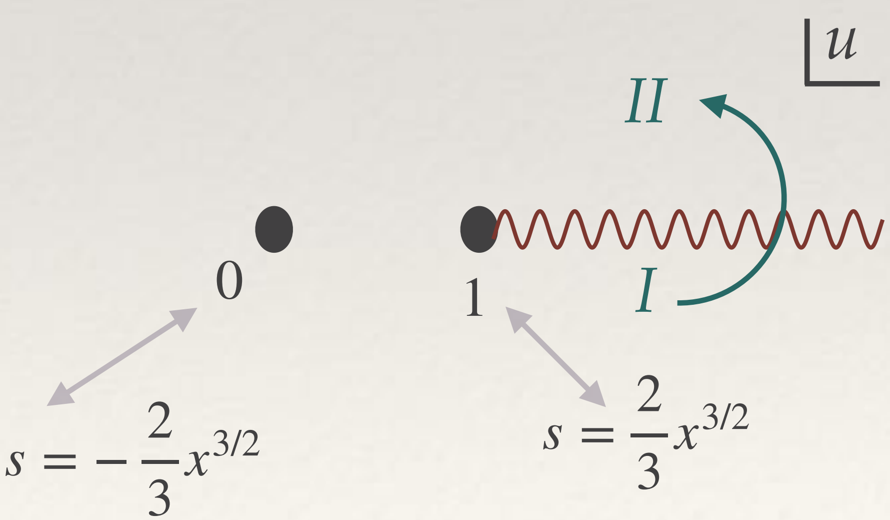
School of Mathematics, Edinburgh University, Edinburgh, United Kingdom.

Connection formula (Stokes phenomenon):

$${}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};u+i\epsilon\right) - {}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};u-i\epsilon\right) = i\,(1+u)^{1/2}\,(u-1)^{-1/2}\,{}_2F_1\left(\frac{5}{6},\frac{1}{6},\frac{1}{2};1-u\right),\qquad u>1$$

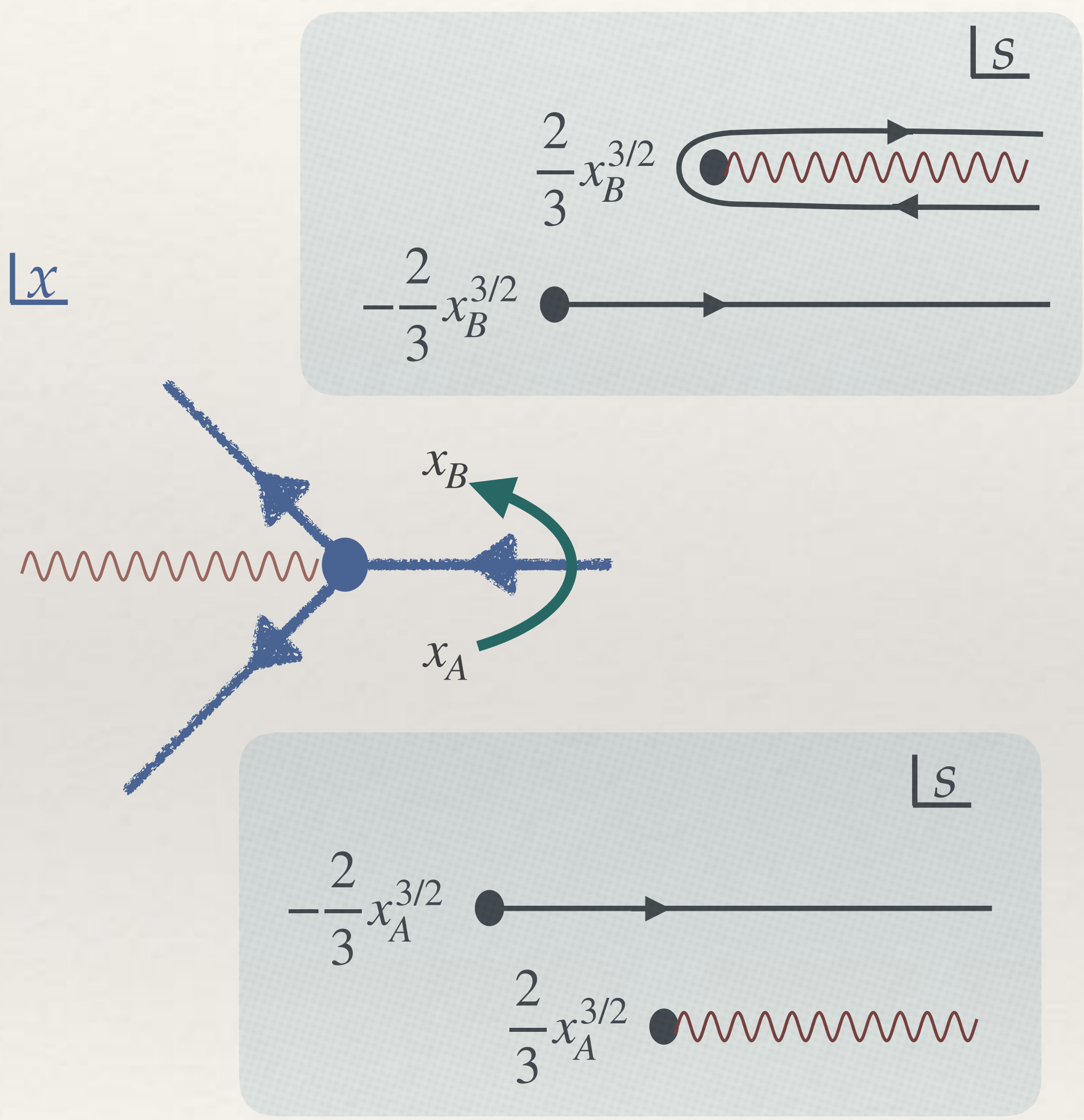
I ⟶ *II*

$$\mathscr{B}\psi_+(s) \to \mathscr{B}\psi_+(s) + i\mathscr{B}\psi_-(s)$$
$$\mathscr{B}\psi_-(s) \to \mathscr{B}\psi_-(s)$$



$$\Delta_{\frac{2}{3}x^{3/2}}\psi_+ = i\psi_-,\quad \Delta_s\psi_{=}0$$

Airy equation

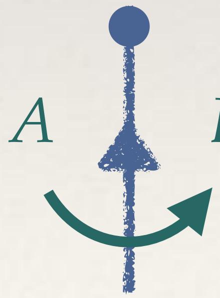


$$\psi_+(x) \to \psi_+(x) \;+\; i\psi_-(x)$$

$$\sim e^{\frac{1}{\hbar}\frac{2x^{3/2}}{3}} \qquad \sim e^{\frac{1}{\hbar}\frac{2x^{3/2}}{3}} \qquad \sim e^{-\frac{1}{\hbar}\frac{2x^{3/2}}{3}}$$

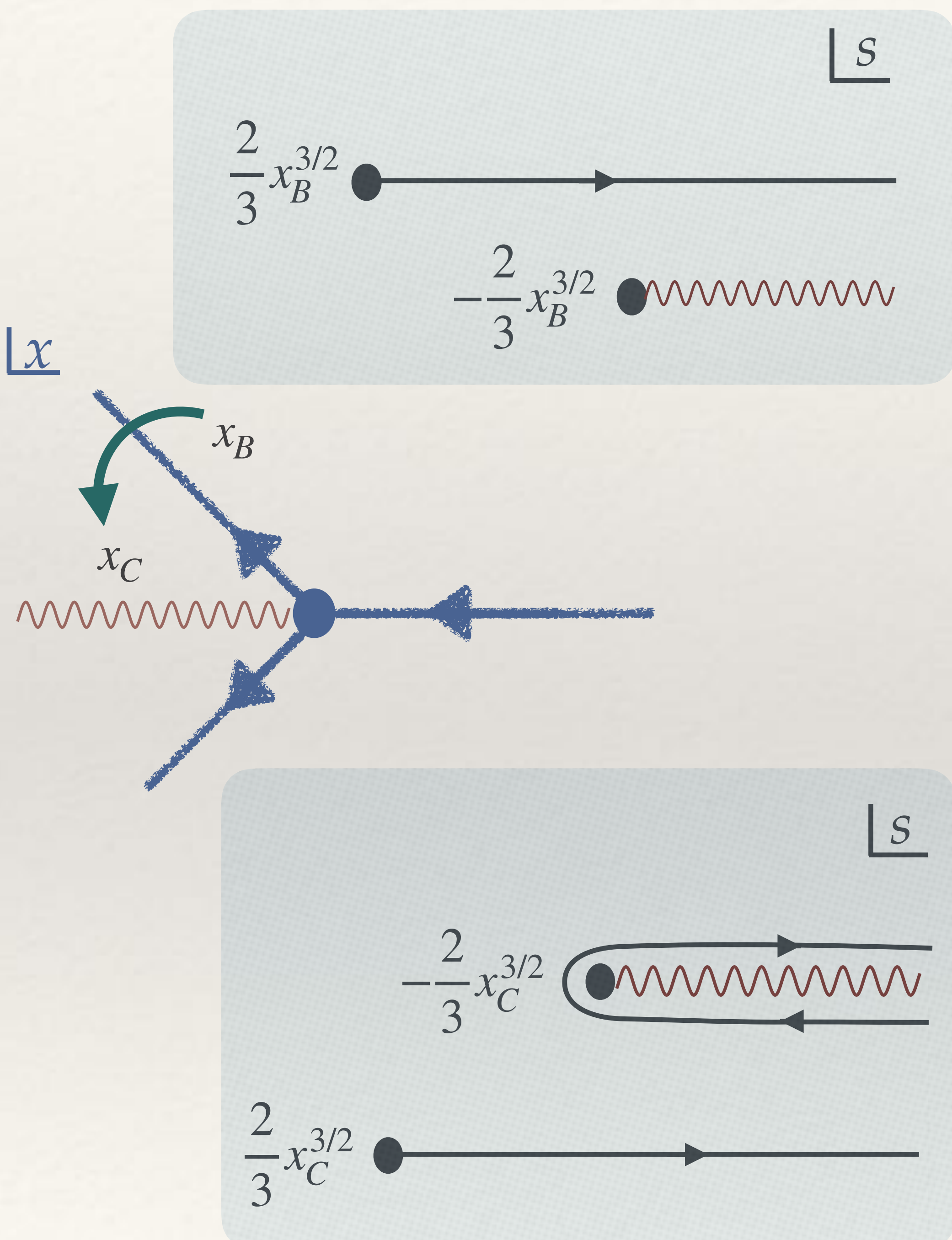
$$\psi_-(x) \to \psi_-(x)$$

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}_B = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}_A$$



$$: \; M_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$$

Airy equation

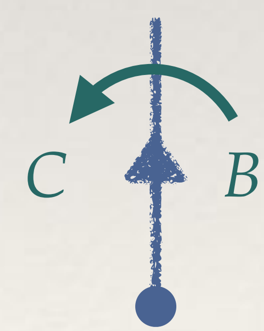


$$\psi_-(x) \to \psi_-(x) + i\psi_+(x)$$

$$\sim e^{\frac{1}{\hbar}\frac{2|x|^{3/2}}{3}} \qquad \sim e^{\frac{1}{\hbar}\frac{2|x|^{3/2}}{3}} \qquad \sim e^{-\frac{1}{\hbar}\frac{2|x|^{3/2}}{3}}$$

$$\psi_+(x) \to \psi_+(x)$$

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}_C = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}_B$$



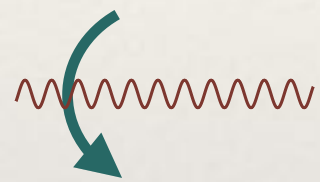
$$: \; M_o = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

Airy equation

$$\psi(x;\hbar) = \sqrt{\frac{\hbar}{P_{even}(x;\hbar)}}\, e^{\pm\frac{1}{\hbar}\int_{x_0}^{x} P_{even}(x;\hbar)dx}$$

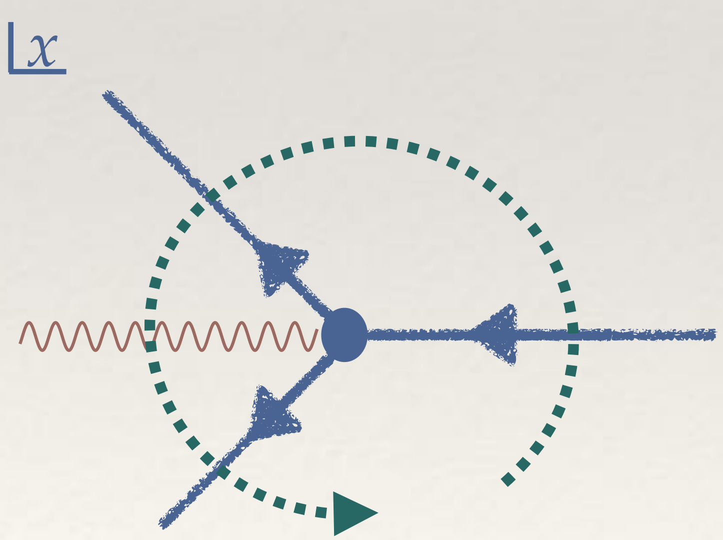
Crossing the branch cut:

$$\sqrt{Q(x)} \to -\sqrt{Q(x)} \quad \Rightarrow \quad P_{even}(x;\hbar) \to -P_{even}(x;\hbar)$$



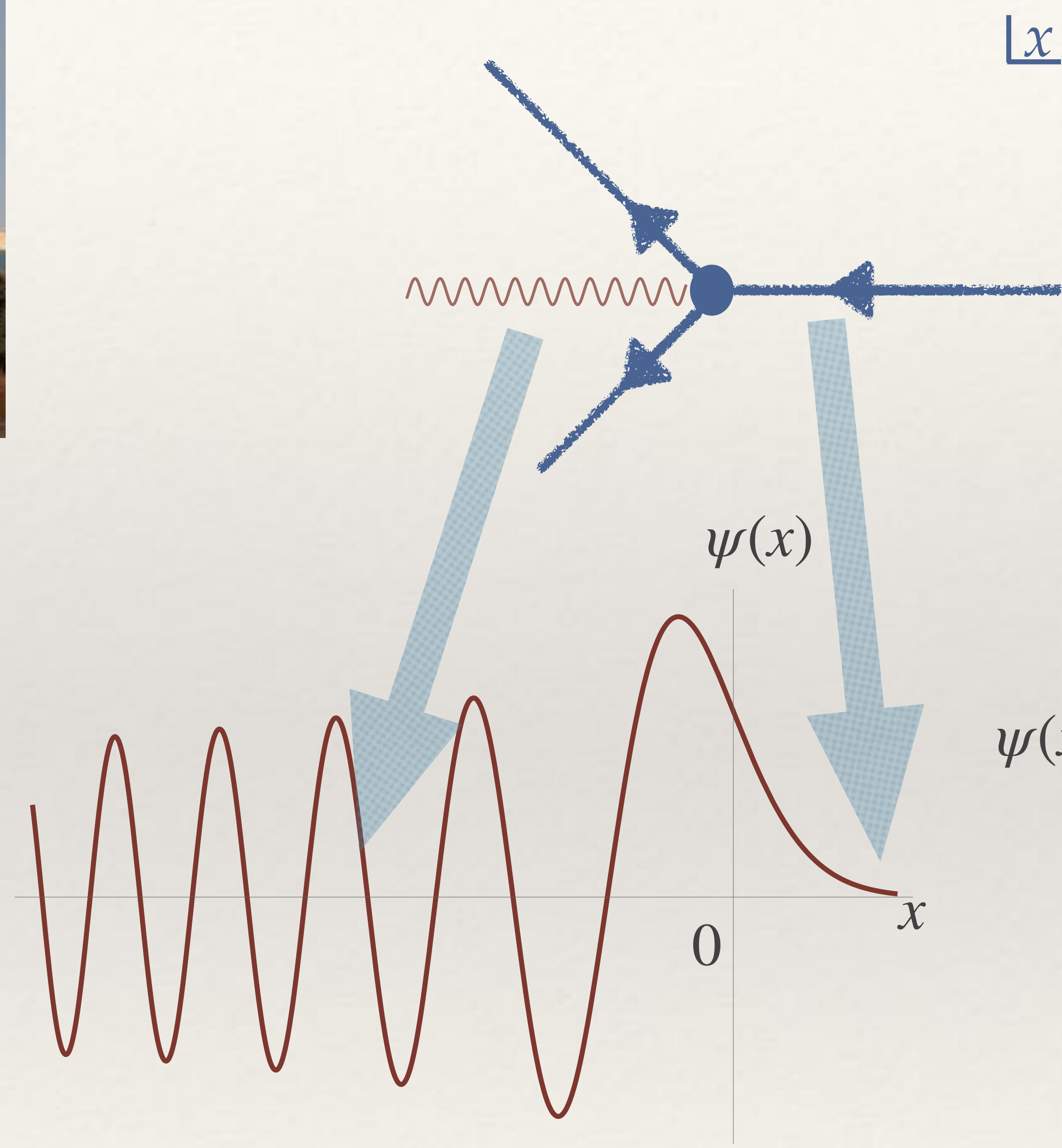
$$M_{br} \equiv \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Monodromy:



check: $M_i M_o M_{br} M_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Airy equation

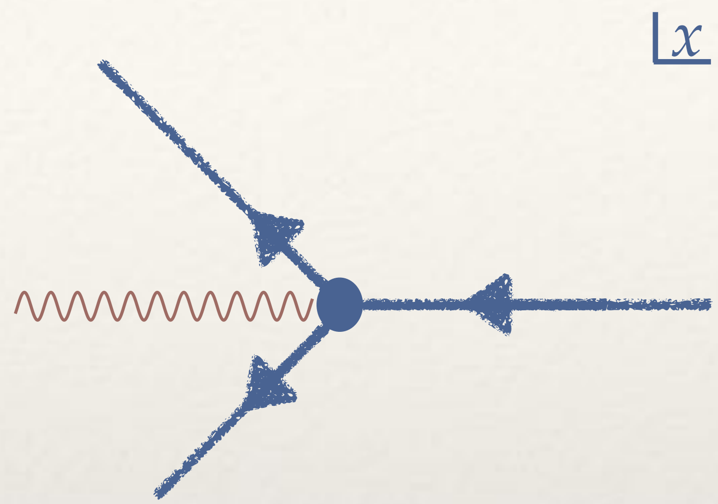


$$\psi(x) = c_+\psi + c_-\psi_-$$

$$\sim \cos\left(\frac{2}{3}|x|^{2/3} - \frac{\pi}{4}\right)$$

$$\psi(x) = c_-\psi_- \sim e^{-\frac{2}{3}x^{2/3}}$$

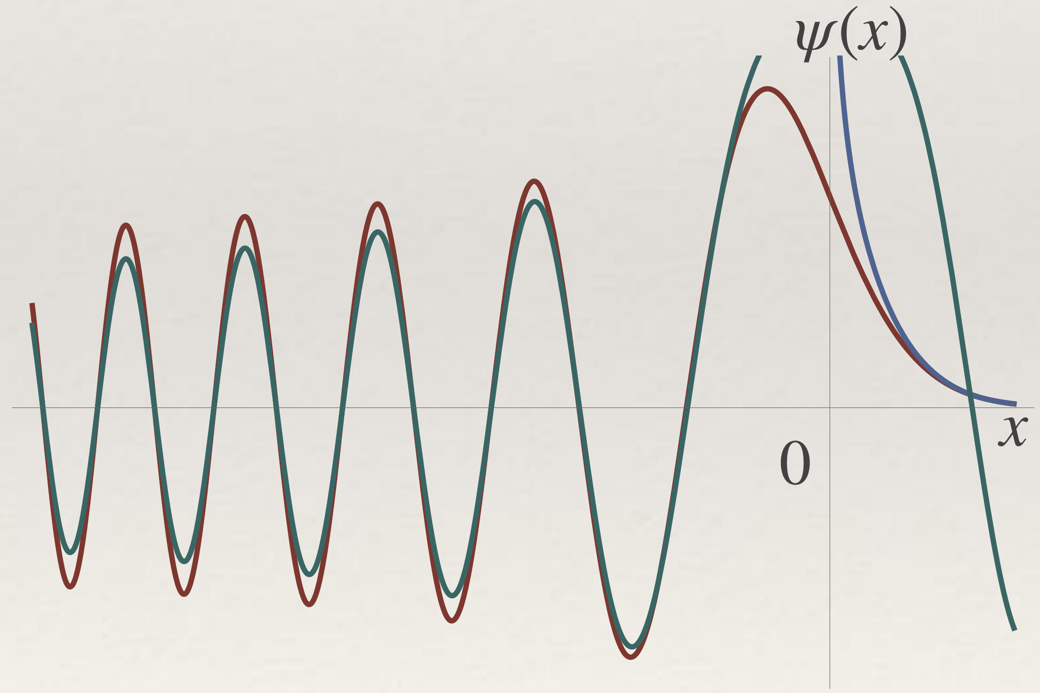
Airy equation



$$\psi(x) = c_-\psi_- \sim e^{-\frac{2}{3}x^{2/3}}$$

$$\psi(x) = c_+\psi + c_-\psi_-$$

$$\sim \cos\left(\frac{2}{3}|x|^{2/3} - \frac{\pi}{4}\right)$$



From local to global analysis

Outlook for lecture II

- In general there are multiple turning points.

- Around each turning point we have local solutions of the form

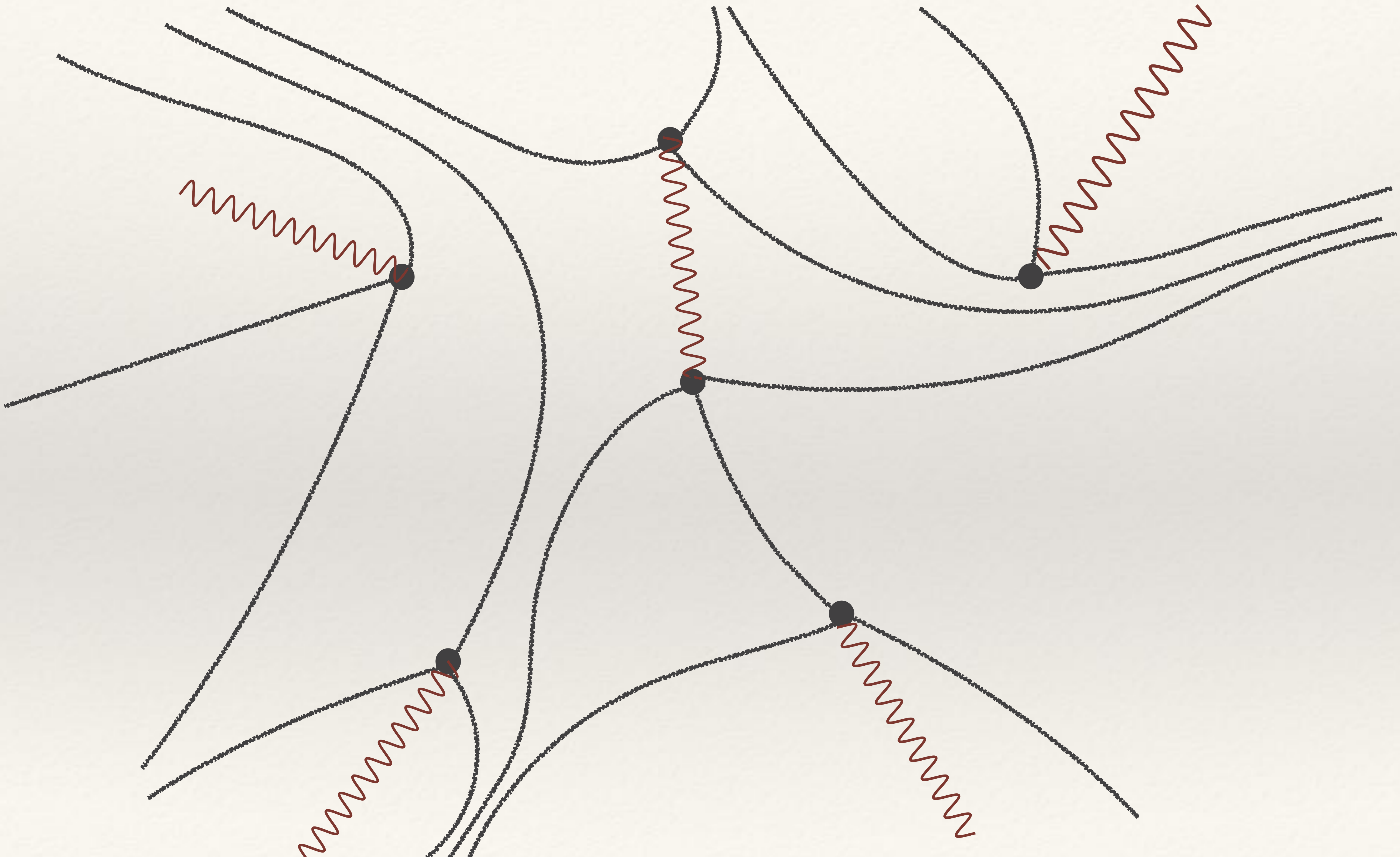
$$c_+\psi_+ + c_-\psi_-$$

where c_+, c_- are resurgent functions (x independent) and uniquely determined once the branches for p are chosen.

- Globally we have resurgent functions that are solutions of the Schrödinger equation and depend analytically on x, constructed by gluing the cs obtained from different turning points

$$\psi \sim c_+\psi_+ + c_-\psi_-$$

From local to global analysis



End of Lecture I