WKB, Eigenvalue Problems and Quantisation in QM

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Spring school on asymptotic methods and applications

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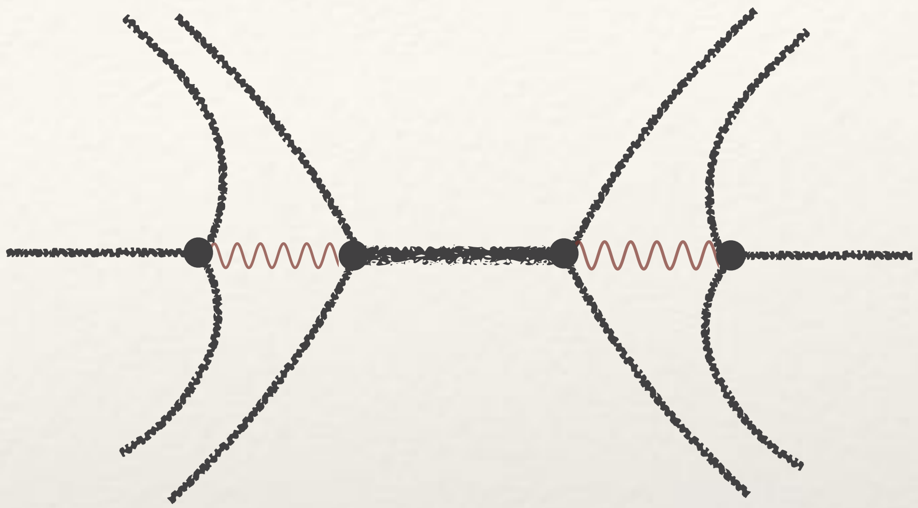
Recap of lecture I

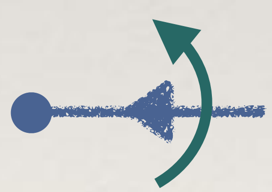
- For for each WKB problem we can draw a Stokes graph

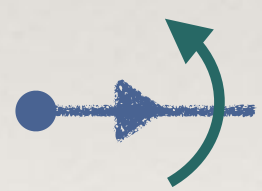


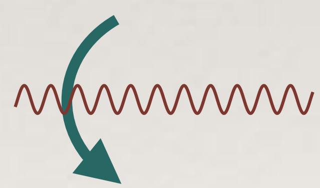
- In each Stokes region the solution is a Borel summable resurgent expression: $c_+\psi_+ + c_-\psi_-$

- Around each (simple) turning point the Stokes jumps are Airy type

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \to M \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

 $: \; M_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$

 $: \; M_o = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$

 $M_{br} \equiv \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$

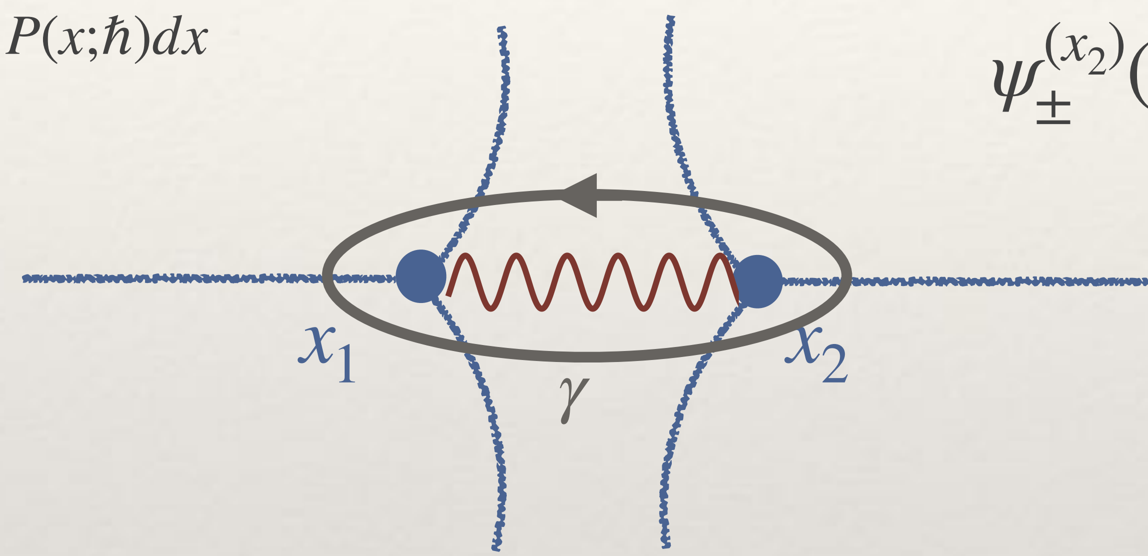
(for $\psi_\pm$ normalized at the turning point)

- Goal: glue these local solutions together

Voros Symbols

Different normalizations

$$\psi_{\pm}^{(x_1)}(x;\hbar) \sim e^{\pm\frac{1}{\hbar}\int_{x_1}^{x} P(x;\hbar)dx} \qquad \psi_{\pm}^{(x_2)}(x;\hbar) \sim e^{\pm\frac{1}{\hbar}\int_{x_2}^{x} P(x;\hbar)dx}$$



$$\psi(x) := c_{-}^{(x_1)}\psi_{-}^{(x_1)} + c_{+}^{(x_1)}\psi_{+}^{(x_1)} = c_{-}^{(x_2)}\psi_{-}^{(x_2)} + c_{+}^{(x_2)}\psi_{+}^{(x_2)}$$

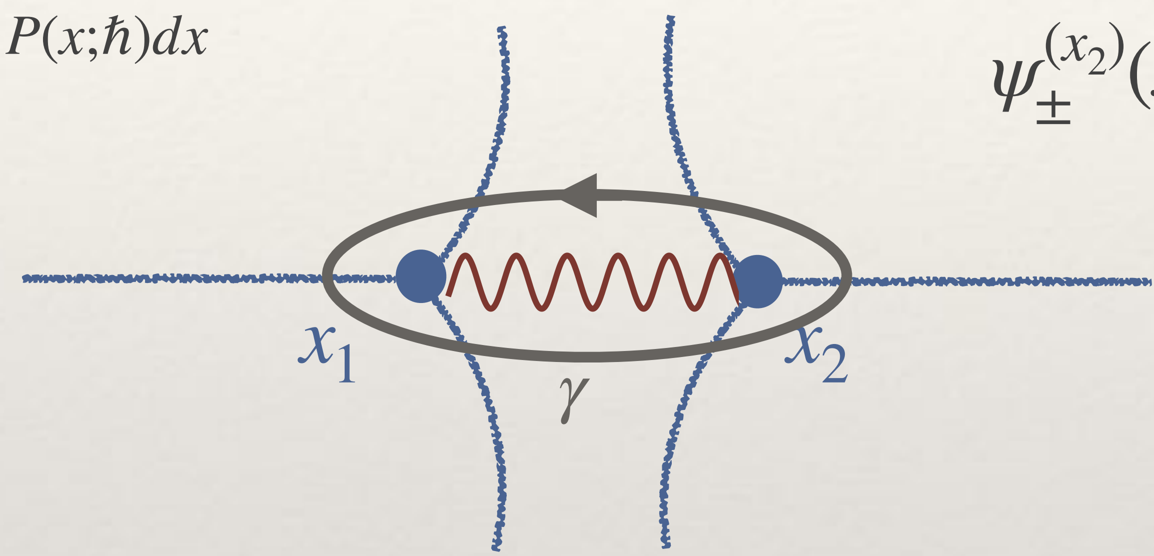
$$\psi_{\pm}^{(x_2)}(x;\hbar) = e^{\pm\frac{1}{\hbar}\int_{x_2}^{x_1} P(x;\hbar)dx}\psi_{\pm}^{(x_1)}(x;\hbar) \qquad \begin{pmatrix} c_+ \\ c_- \end{pmatrix}_{(x_1)} = \begin{pmatrix} e^{\frac{1}{\hbar}\int_{x_2}^{x_1} Pdx} & 0 \\ 0 & e^{-\frac{1}{\hbar}\int_{x_2}^{x_1} Pdx} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}_{(x_2)}$$

Voros Symbols

Different normalizations

$$\psi_\pm^{(x_1)}(x;\hbar) \sim e^{\pm\frac{1}{\hbar}\int_{x_1}^{x} P(x;\hbar)dx}$$

$$\psi_\pm^{(x_2)}(x;\hbar) \sim e^{\pm\frac{1}{\hbar}\int_{x_2}^{x} P(x;\hbar)dx}$$



$$\int_{x_2}^{x_1} P(x;\hbar)dx = \frac{1}{2}\oint_\gamma P(x;\hbar)dx$$

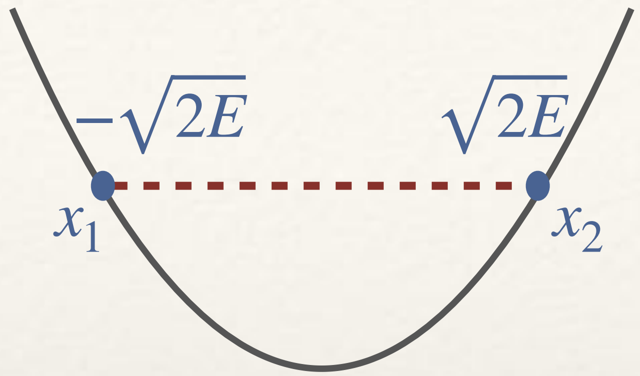
$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix}_{(x_1)} = \begin{pmatrix} \mathscr{V}_\gamma^{1/2} & 0 \\ 0 & \mathscr{V}_\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}_{(x_2)}$$

$$\mathscr{V}_\gamma := e^{\frac{1}{\hbar}\oint_\gamma P(x;\hbar)dx} = e^{\frac{1}{\hbar}S_\gamma(E)}$$

Voros symbol

Harmonic oscillator

$$\left(-\hbar^2\frac{d^2}{dx^2}+x^2-2E\right)\psi(x)=0$$

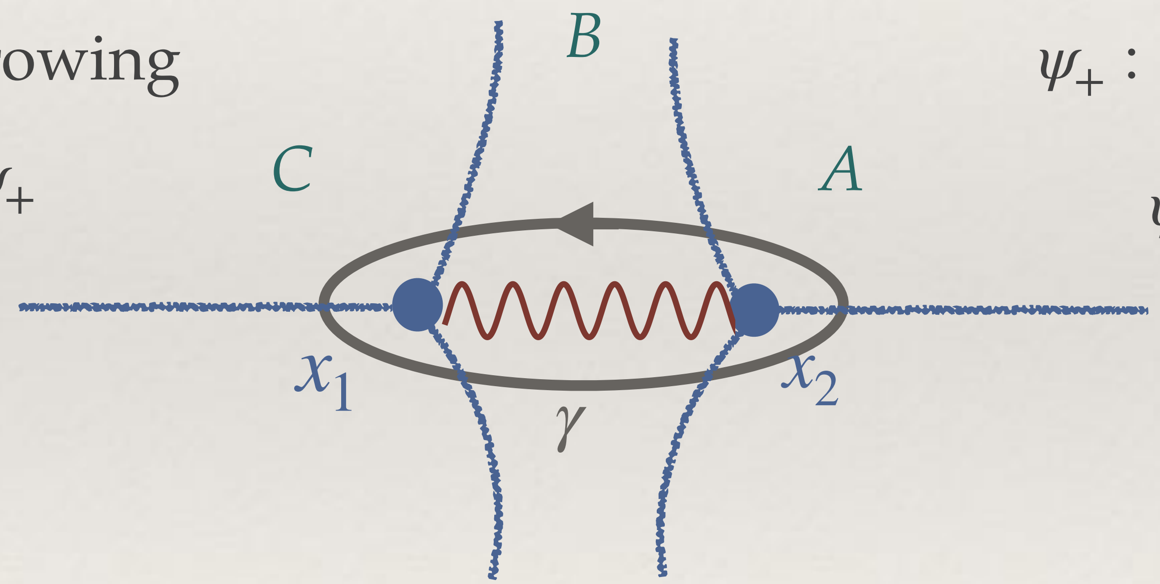


ψ_- : exp. growing

ψ_+ : exp. growing

$\psi \sim c_+^{(C)}\psi_+$

$\psi \sim c_-^{(A)}\psi_-$



Stokes jumps: relate $c_\pm^C$ with $c_\pm^A$

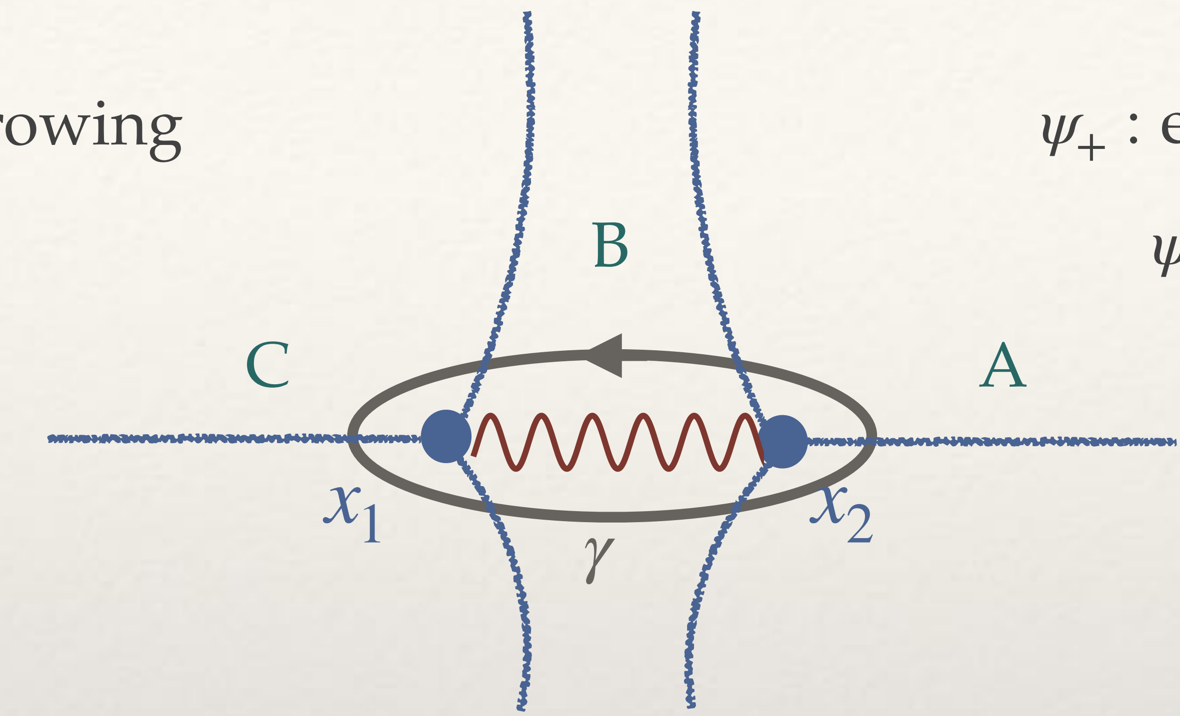
Harmonic oscillator

ψ_- : exp. growing

$\psi \sim \psi_+$

ψ_+ : exp. growing

$\psi \sim \psi_-$



$c_+^A = 0$

$$\begin{pmatrix} c_+^C \\ c_-^C \end{pmatrix}_{(x_2)} = \begin{pmatrix} \mathscr{V}_\gamma^{1/2} & 0 \\ 0 & \mathscr{V}_\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathscr{V}_\gamma^{-1/2} & 0 \\ 0 & \mathscr{V}_\gamma^{1/2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_+^A \\ c_-^A \end{pmatrix}_{(x_2)}$$

$c_+^C = i(1 + \mathscr{V}_\gamma) c_-^A$ ⟶ For ψ to be square integrable:

$$1 + \mathscr{V}_\gamma = 0$$

Voros-Silverstone connection formula

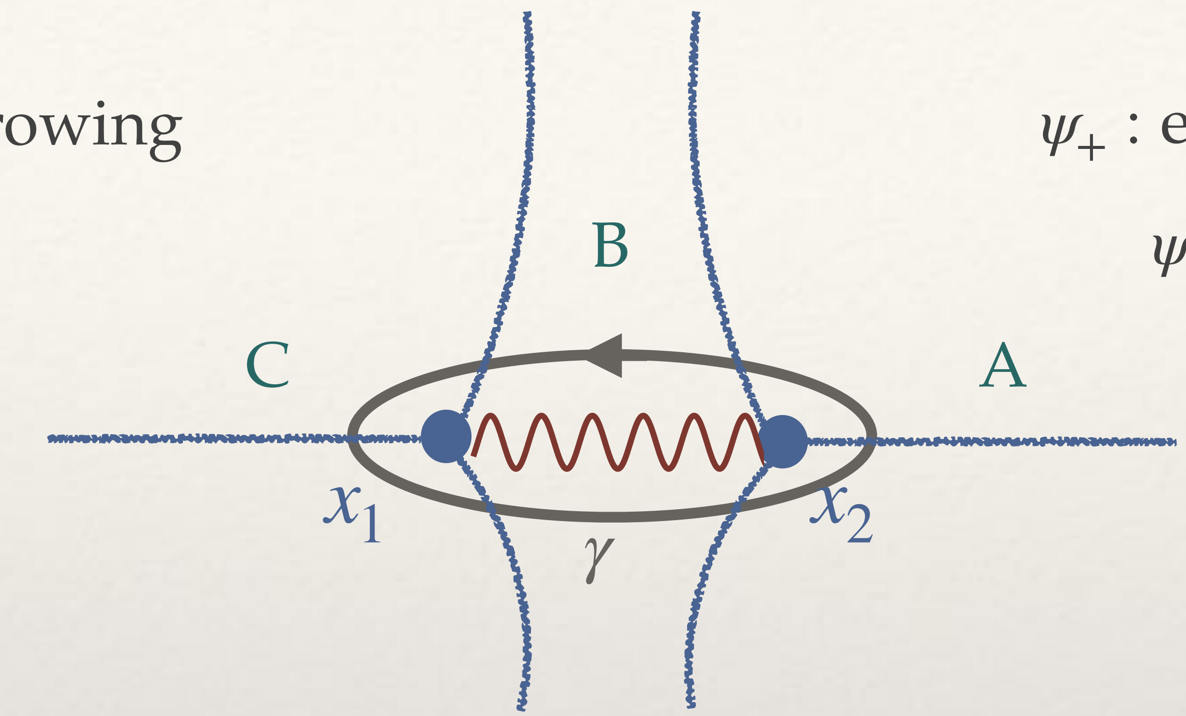
Harmonic oscillator

ψ_- : exp. growing

$\psi \sim \psi_+$

ψ_+ : exp. growing

$\psi \sim \psi_-$



$c_+^A = 0$

$$\begin{pmatrix} c_+^C \\ c_-^C \end{pmatrix}_{(x_2)} = \begin{pmatrix} \mathscr{V}_\gamma^{1/2} & 0 \\ 0 & \mathscr{V}_\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathscr{V}_\gamma^{-1/2} & 0 \\ 0 & \mathscr{V}_\gamma^{1/2} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_+^A \\ c_-^A \end{pmatrix}_{(x_2)}$$

$c_+^C = i(1 + \mathscr{V}_\gamma)c_-^A$ ⟶ For ψ to be square integrable:

$$1 + \mathscr{V}_\gamma = 0$$

Voros-Silverstone connection formula [Voros, ’83, Silverstone ’85]

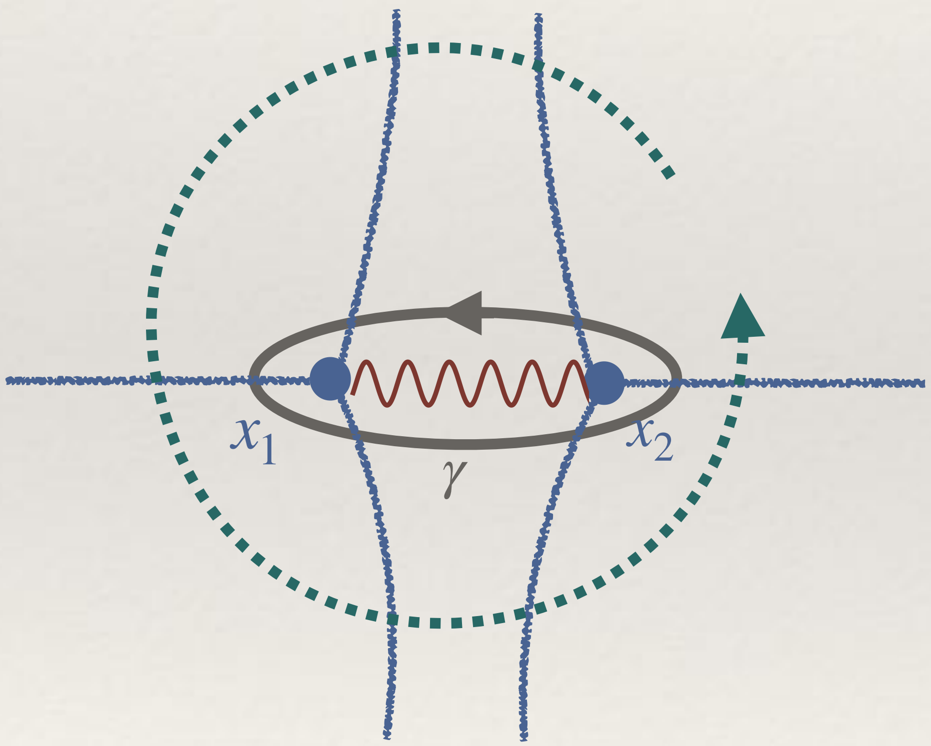
Harmonic oscillator

As a cross check lets see what happens when we go around a full circle

Monodromy:

$$\begin{pmatrix} \psi_+^{(x_2)} \\ \psi_-^{(x_2)} \end{pmatrix} \to \begin{pmatrix} \mathcal{V}_\gamma^{1/2} & \\ 0 & \mathcal{V}_\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_\gamma^{1/2} & \\ 0 & \mathcal{V}_\gamma^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_+^{(x_2)} \\ \psi_-^{(x_2)} \end{pmatrix}$$

Exercise!



$$= -\begin{pmatrix} \mathcal{V}_\gamma & \\ 0 & \frac{1}{\mathcal{V}_\gamma} \end{pmatrix} \begin{pmatrix} \psi_+^{(x_2)} \\ \psi_-^{(x_2)} \end{pmatrix}$$

For ψ to be single valued:

$$\mathcal{V}_\gamma = -1$$

Harmonic oscillator: exact quantization

$\mathscr{V}_\gamma(E) + 1 := f(\mathscr{V}_\gamma) = 0$ *is the exact quantization condition!*

$$\mathscr{V}_\gamma = e^{\frac{1}{\hbar}S_\gamma(E)} \qquad S_\gamma(E) = 2\int_{-\sqrt{2E}}^{\sqrt{2E}} \sqrt{x^2 - 2E}\,dx = 2\pi i E$$

Exercise: verify that higher order actions do not contribute

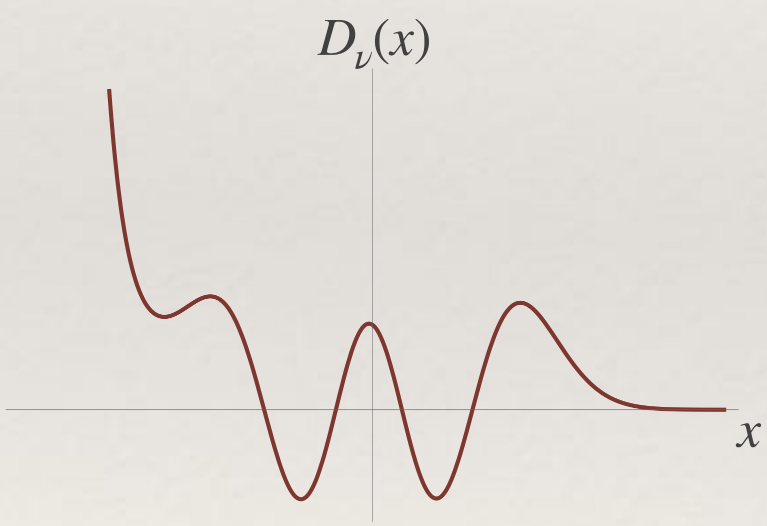
$$e^{\frac{1}{\hbar}S_\gamma(E)} = -1 \Rightarrow E = \hbar(N + 1/2)$$

Harmonic oscillator: exact quantization

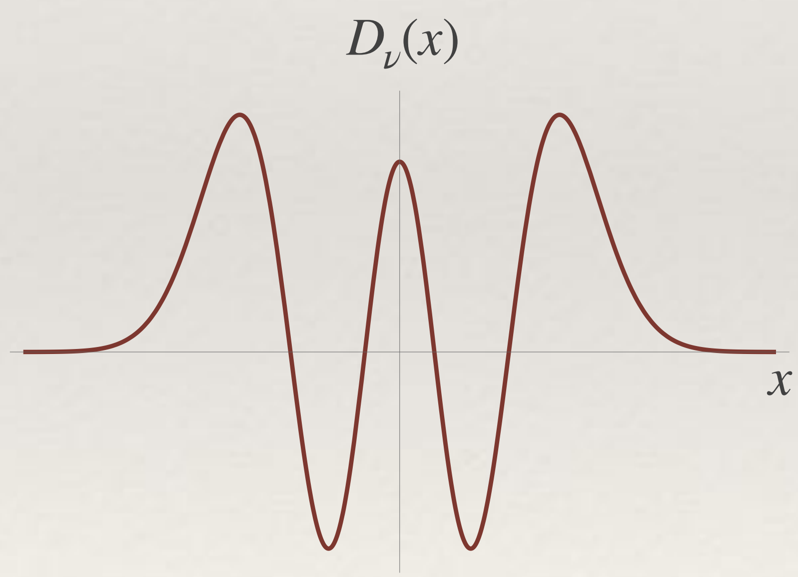
$$e^{\frac{1}{\hbar}S_\gamma(E)} = -1 \Rightarrow E = \hbar(N + 1/2)$$

The solutions of the Schrödinger equation (Weber equation) are parabolic cylinder functions

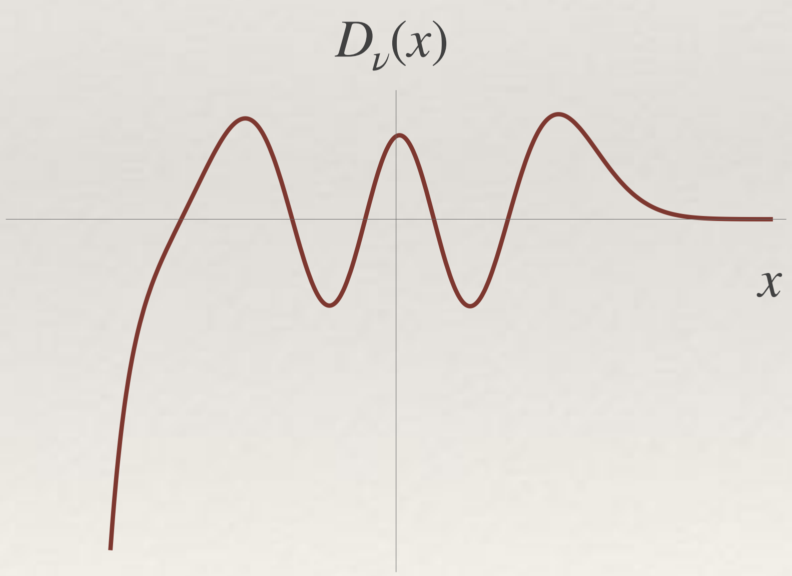
$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}x^2 - \nu - \frac{1}{2}\right) D_\nu(x) = 0 \qquad \nu := \frac{E}{\hbar} - \frac{1}{2}$$



$\nu = 3.9$



$\nu = 4$



$\nu = 4.1$

Exact quantization

$$S(E) = \oint \sqrt{2(V-E)}dx = 2\pi i\hbar N$$

Bohr-Sommerfeld quantization condition

↑

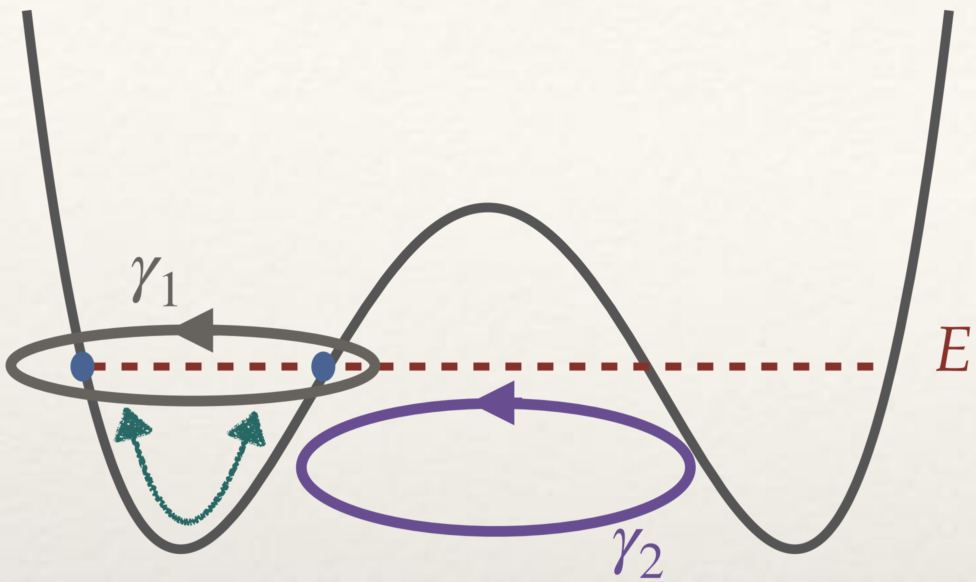
Classical phase space volume

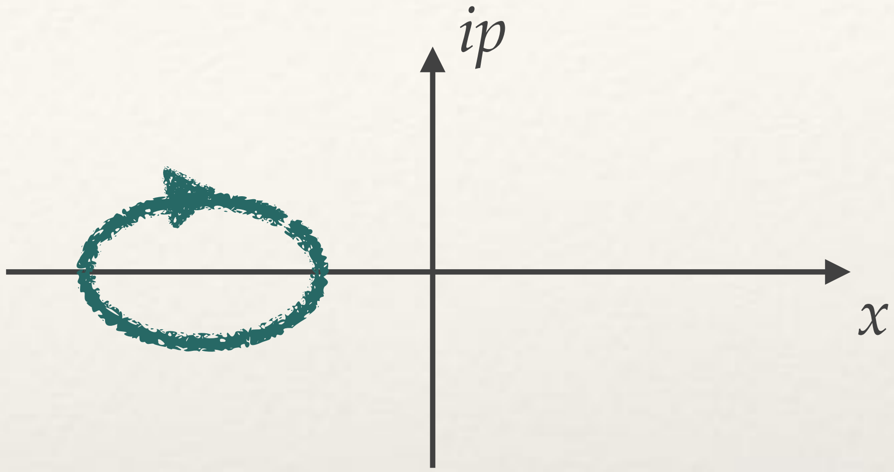
$$S(E) = \oint P dx = \oint \left(\sqrt{2(V-E)} + \hbar^2 P_2(x) + \hbar^2 P_4(x) \ldots \right) \overset{?}{=} 2\pi i N$$

Due to its asymptotic character, the series in Eq. (16) diverges for actual, finite values of B, and it can therefore be used to find energy levels only to a limited degree of approximation. The quality of the approximation is, in general, better for systems which are nearly classical than for those showing large quantum effects. This is not surprising since the first term of the energy level equation gives the classical Bohr theory.

[Dunham, 1932]

Exact quantization



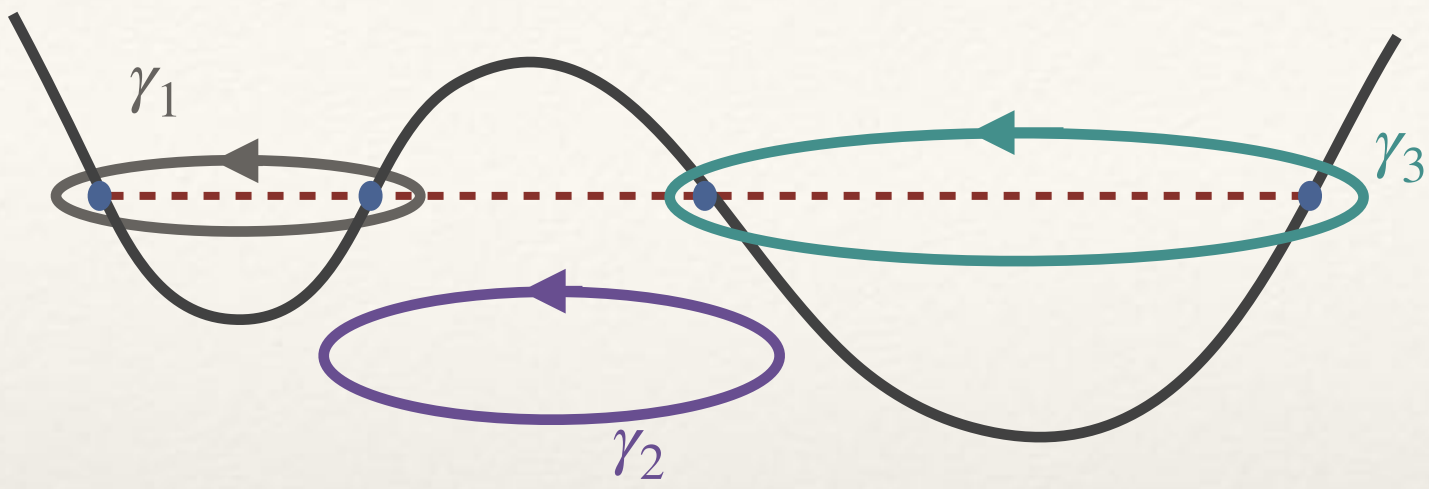


$$S_{\gamma_1}(E) = \oint_{\gamma_1} P dx = \oint_{\gamma_1} \left(\sqrt{2(V-E)} + \hbar^2 P_2(x) + \hbar^2 P_4(x) \dots \right) \stackrel{?}{=} 2\pi i N$$

Not quite!

- Leaves out "non perturbative" parts of the resurgent expansion (i.e. $e^{-\frac{C}{\hbar}}$ terms).
- They are associated with tunneling and contribute to the energy spectrum.
- Have to take into account the "tunneling action" $\left(C = \frac{1}{2} \oint_{\gamma_2} P dx \right)$

Voros Symbols and exact WKB



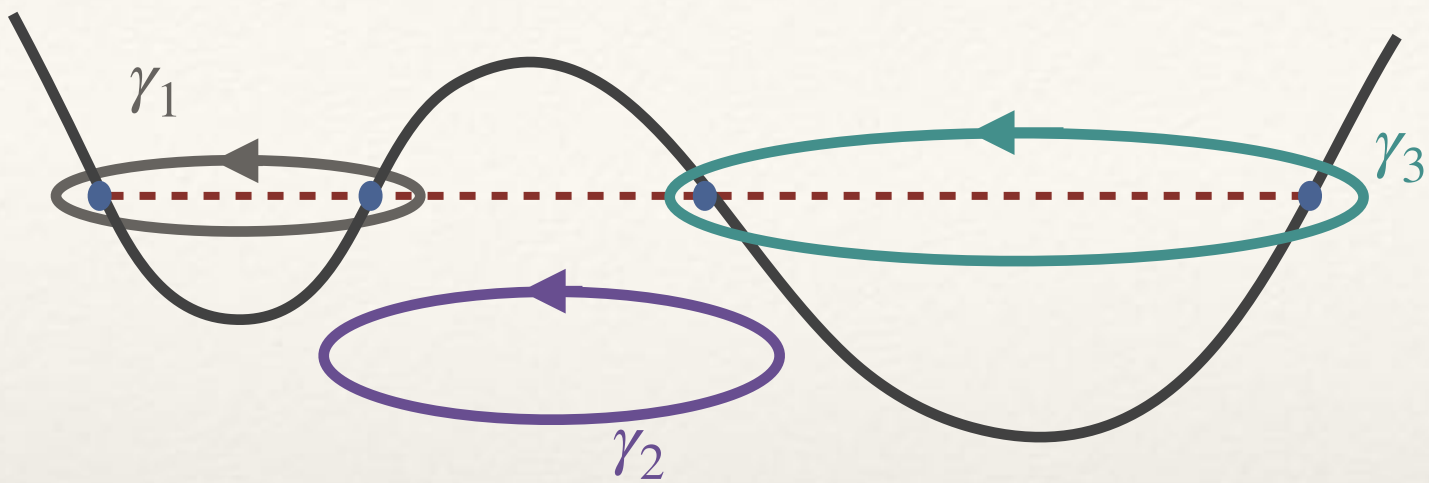
$$\mathscr{V}_{\gamma_i} := e^{\frac{1}{\hbar}\oint_{\gamma_i} P(x;\hbar)dx}, \quad \gamma_i \in H_1(\Sigma)$$

- Voros symbols are resurgent functions of $\hbar$ and they exhibit Stokes phenomena which relates different Voros symbols

- Exact quantization condition can be expressed implicitly as

$$f(\mathscr{V}_{\gamma_1}, \mathscr{V}_{\gamma_2}, \mathscr{V}_{\gamma_3}, \ldots) = 0$$

Voros Symbols and Stokes phenomena



$$\mathscr{V}_{\gamma_i} := e^{\frac{1}{\hbar}\oint_{\gamma_i} P_{even}(x;\hbar)dx}, \quad \gamma_i \in H_1(\Sigma)$$

- $\mathscr{V}_{\gamma_i}$ is not Borel summable if there is a singularity in the path of Borel integral
- This happens when γ intersects another cycle, say γ'.
- We can resolve this Borel ambiguity by $\arg\hbar \to \arg\hbar \pm i\epsilon$

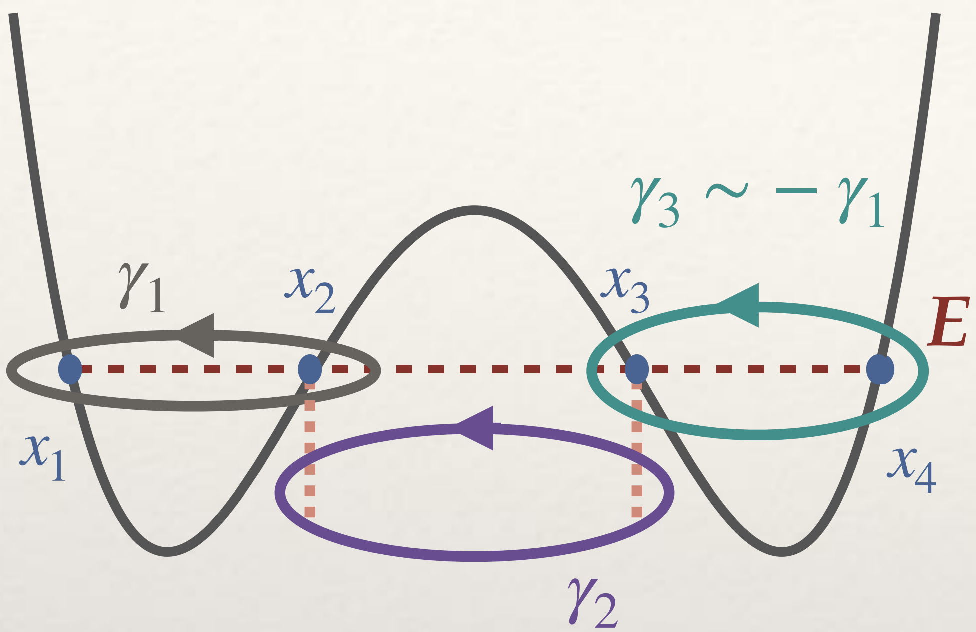
$$\mathcal{S}_{\theta^+}[\mathscr{V}_\gamma] \neq \mathcal{S}_{\theta^-}[\mathscr{V}_\gamma]$$

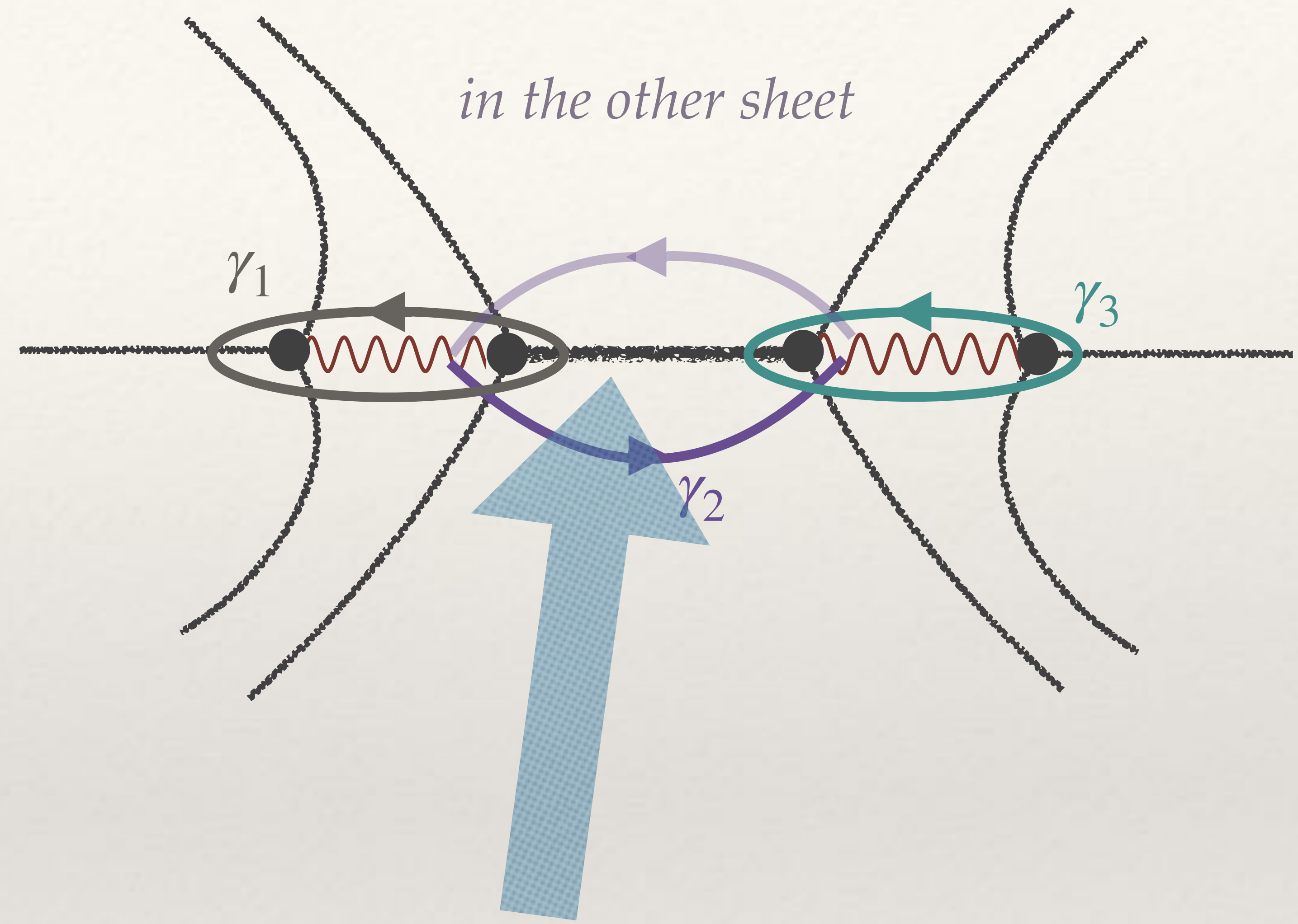
$\theta := \arg\hbar$

Stokes automorphism:

$$\mathfrak{S}[\mathscr{V}_\gamma] := \mathcal{S}_{\theta^+} \circ \mathcal{S}_{\theta^-}^{-1}[\mathscr{V}_\gamma]$$

Double well





Singular Stokes line

[DDP: "critical"]

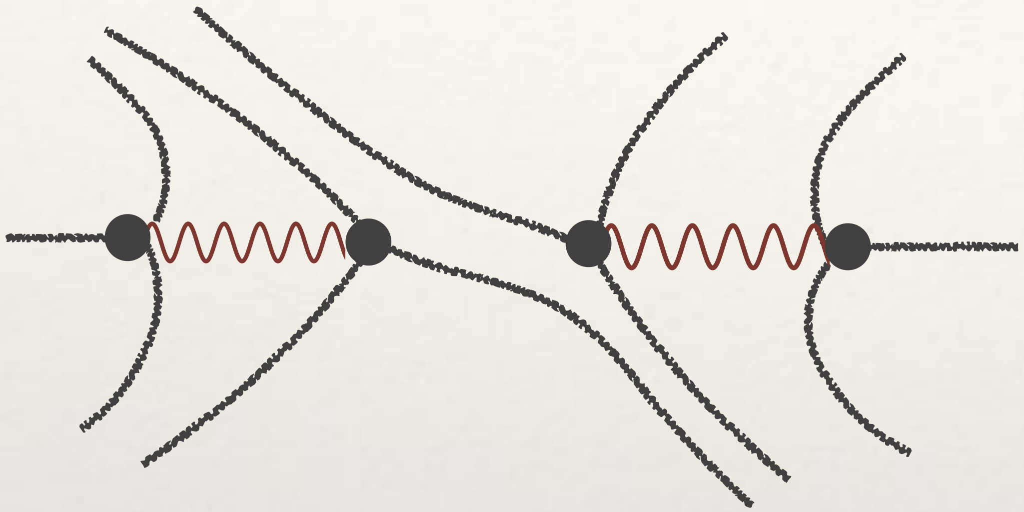
(connects two turning points)

Singularity in Borel plane.

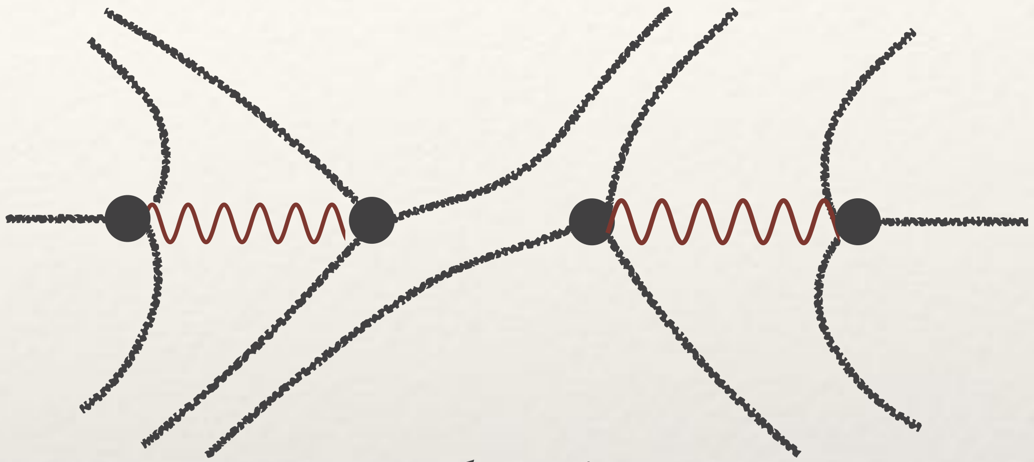
Σ : *g=1* curve

Double well

Resolve the singular Stokes line:



$\arg \hbar = -i\epsilon$



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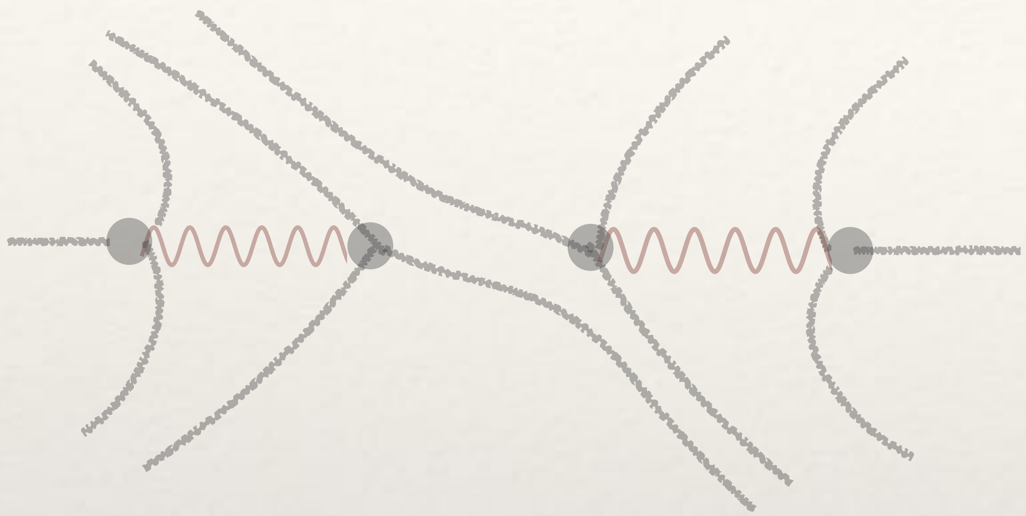

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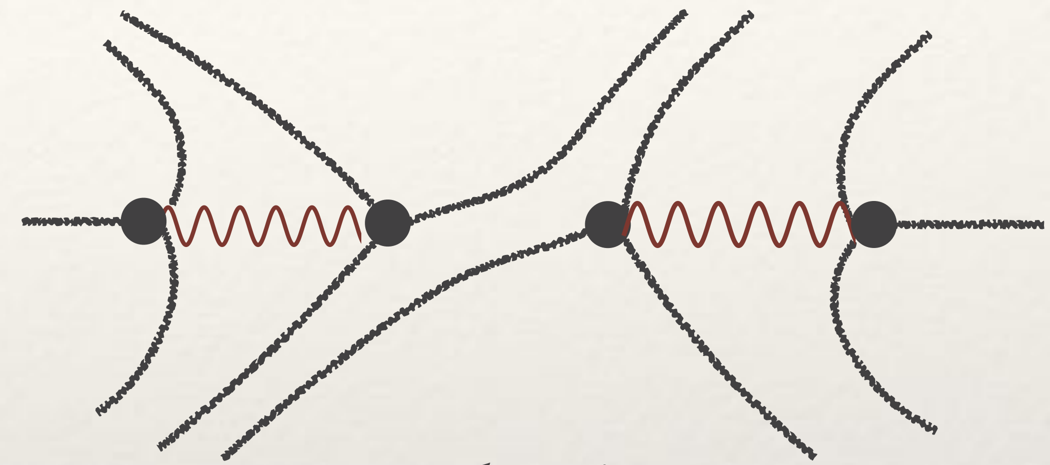
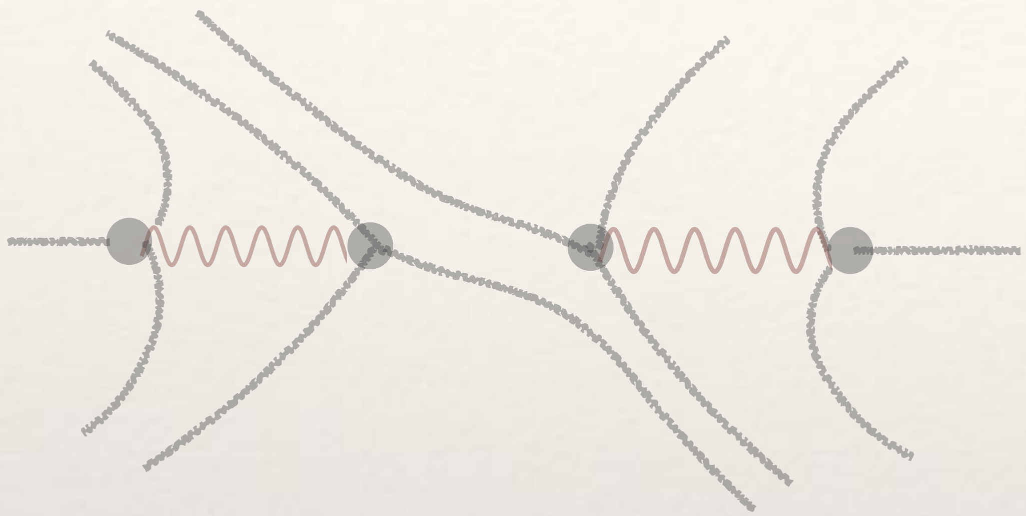


$\arg \hbar = i\epsilon$

choose $\arg \hbar = i\epsilon$

Double well

Resolve the singular Stokes line:



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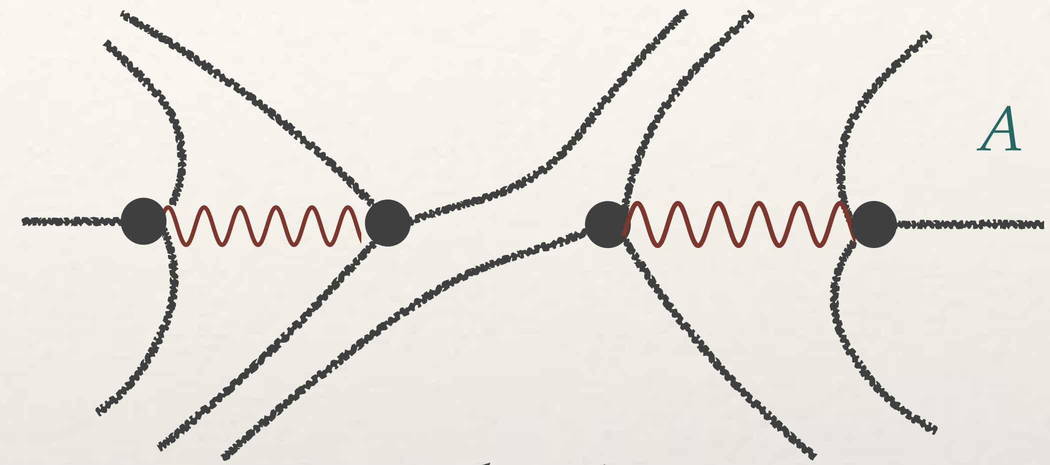


$\arg \hbar = i\epsilon$

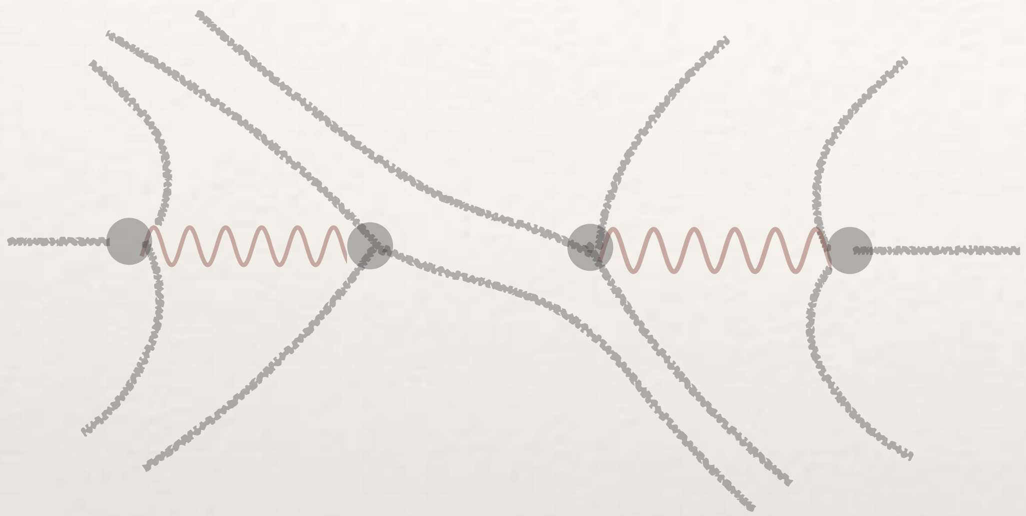
choose $\arg \hbar = i\epsilon$

Boundary conditions:

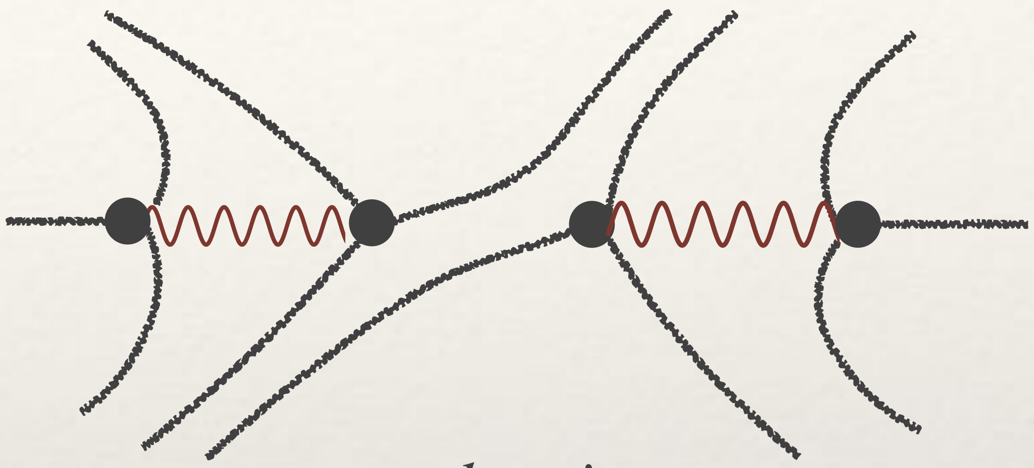
- Demand $\psi \sim \psi_-$ in region A
- Parity $\begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+\eta \\ 1-\eta \end{pmatrix}$ $\quad \eta = 1,\quad$ (parity even) $\quad \eta = -1,\quad$ (parity odd)

Double well

Resolve the singular Stokes line:



$\arg \hbar = -i\epsilon$



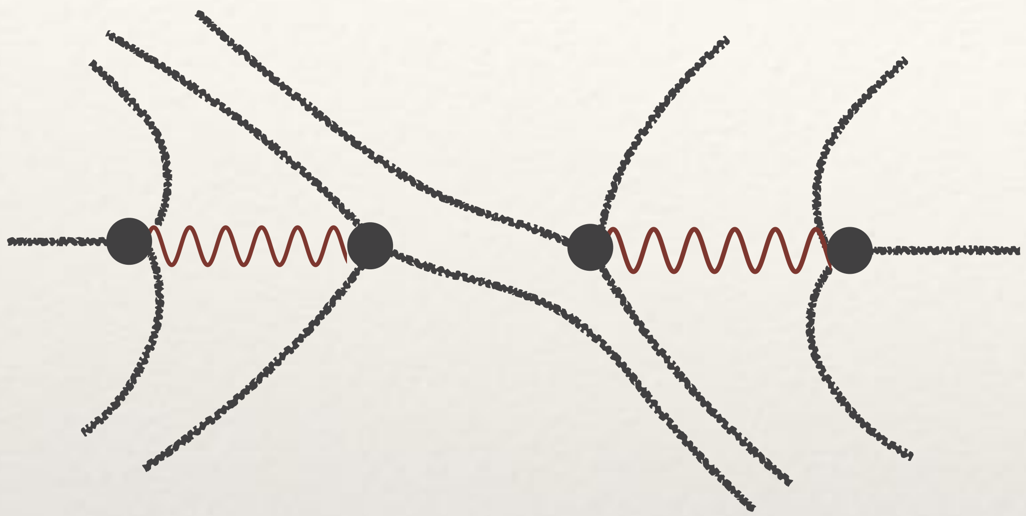
$\arg \hbar = i\epsilon$

Connection formula:

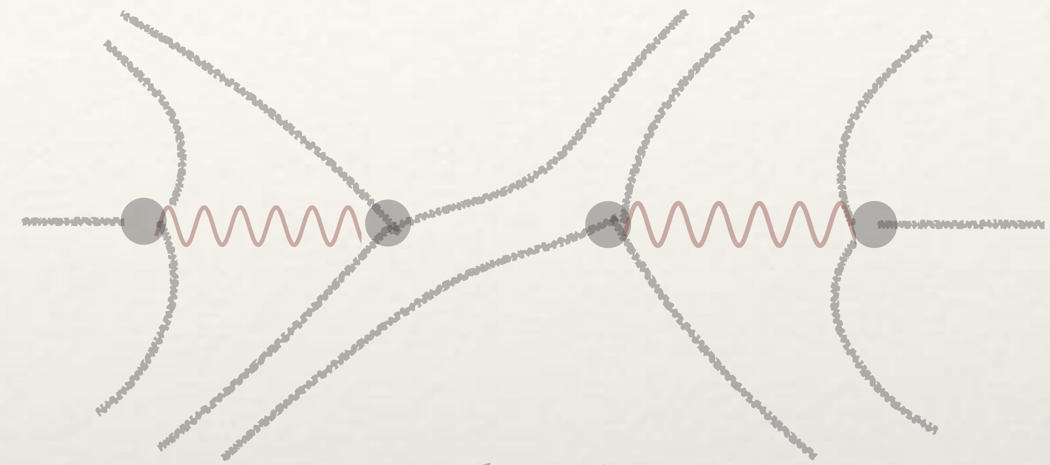
$$\mathcal{V}_{\gamma_1}^{-1} - i\eta \mathcal{V}_{\gamma_2}^{1/2} + 1 = 0$$

Double well

Resolve the singular Stokes line:



$\arg \hbar = - i\epsilon$



$\arg \hbar = i\epsilon$

Connection formula:

$$\mathscr{V}_{\gamma_1} + i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$$

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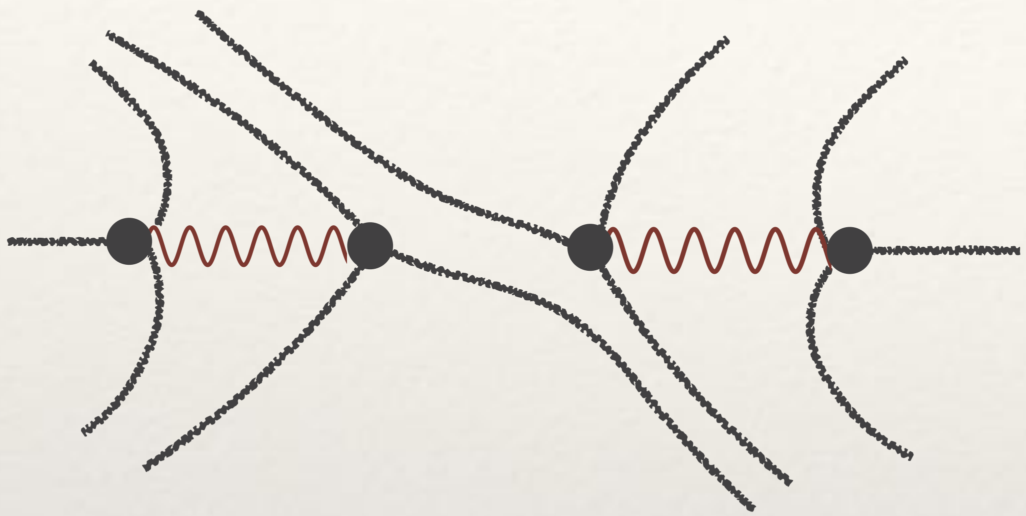
Double well

Resolve the singular Stokes line:



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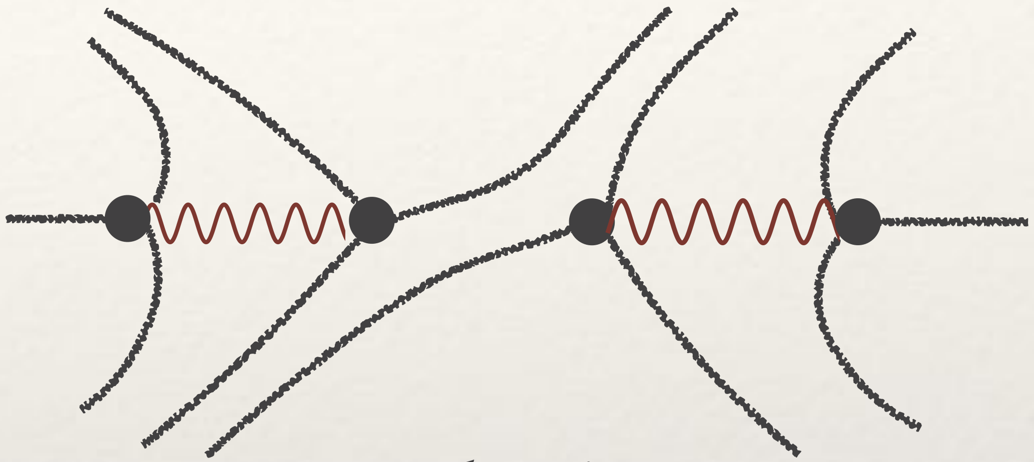


$\arg \hbar = i\epsilon$

Connection formula:

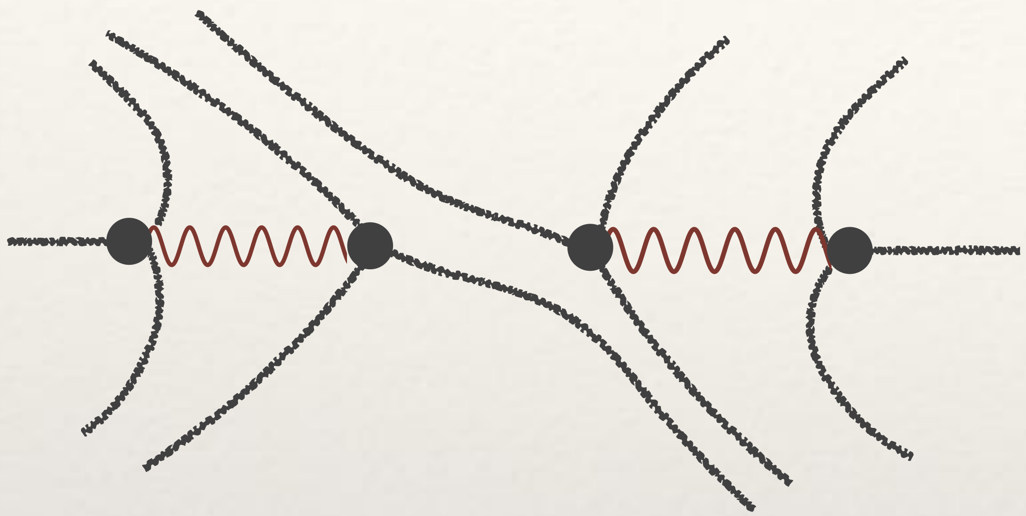
$$\mathscr{V}_{\gamma_1,-} + i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$$

Connection formula:

$$\mathscr{V}_{\gamma_1,+}^{-1} - i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$$

Double well

Resolve the singular Stokes line:



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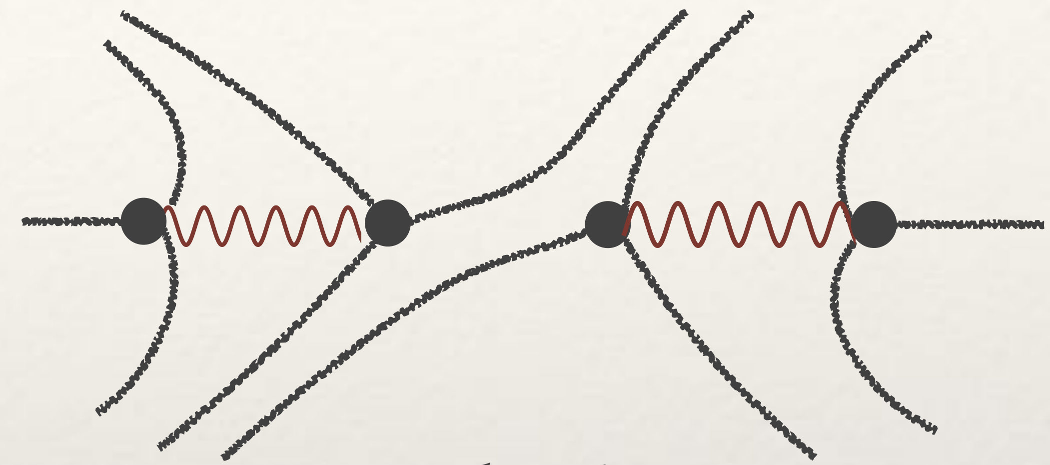


$\arg \hbar = i\epsilon$

Connection formula:

$$2 + \mathscr{V}_{\gamma_1,-} + \mathscr{V}_{\gamma_1,-}^{-1} + \mathscr{V}_{\gamma_1,-}^{-1}\mathscr{V}_{\gamma_2} = 0$$

$$\mathscr{V}_{\gamma_1,+} = \mathscr{V}_{\gamma_1,-}(1 + \mathscr{V}_{\gamma_2})^{-1}$$

$$i.e.: \quad \mathfrak{S}\mathscr{V}_{\gamma_1} = \mathscr{V}_{\gamma_1}(1 + \mathscr{V}_{\gamma_2})^{-1}$$

Connection formula:

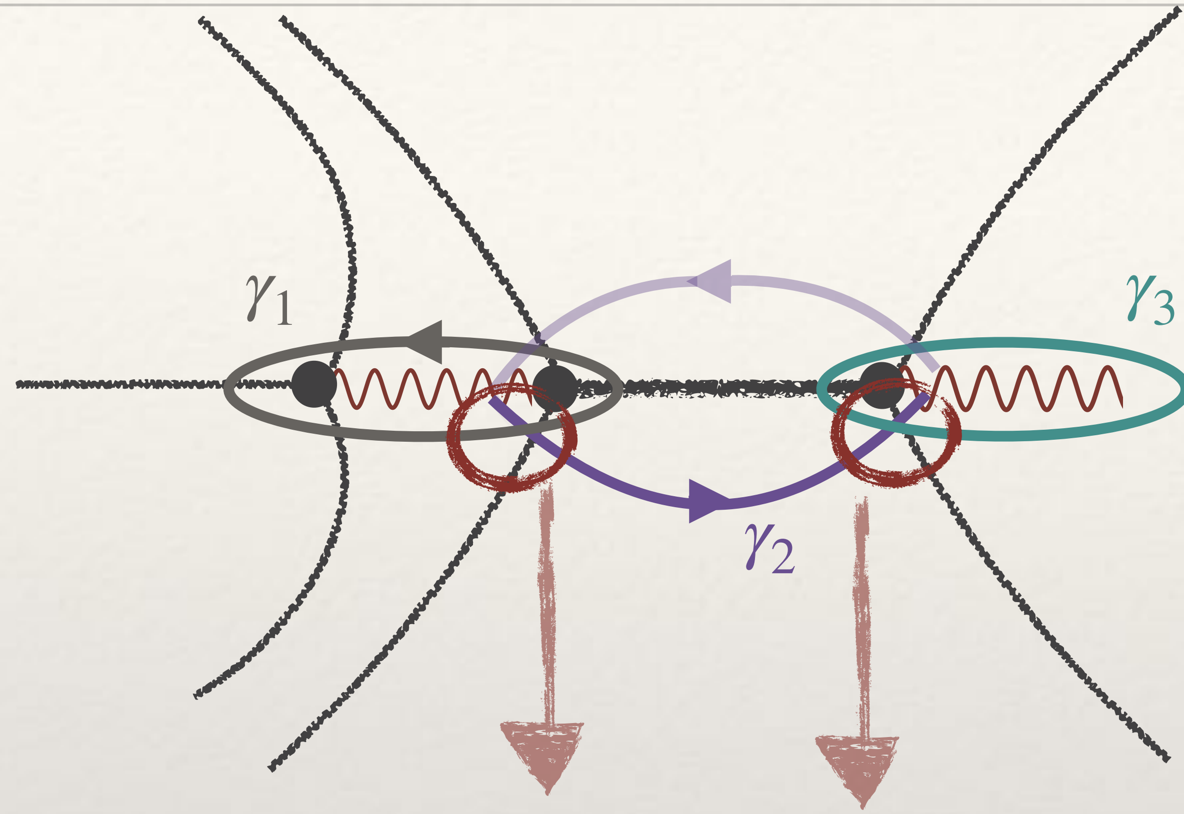
$$2 + \mathscr{V}_{\gamma_1,+} + \mathscr{V}_{\gamma_1,+}^{-1} + \mathscr{V}_{\gamma_1,+}\mathscr{V}_{\gamma_2} = 0$$

$$\mathscr{V}_{\gamma_2,+} = \mathscr{V}_{\gamma_2,-}$$

$$\mathfrak{S}\mathscr{V}_{\gamma_2} = \mathscr{V}_{\gamma_2}$$

Delabaere Dillinger Pham formula

Delabaere Dillinger Pham formula



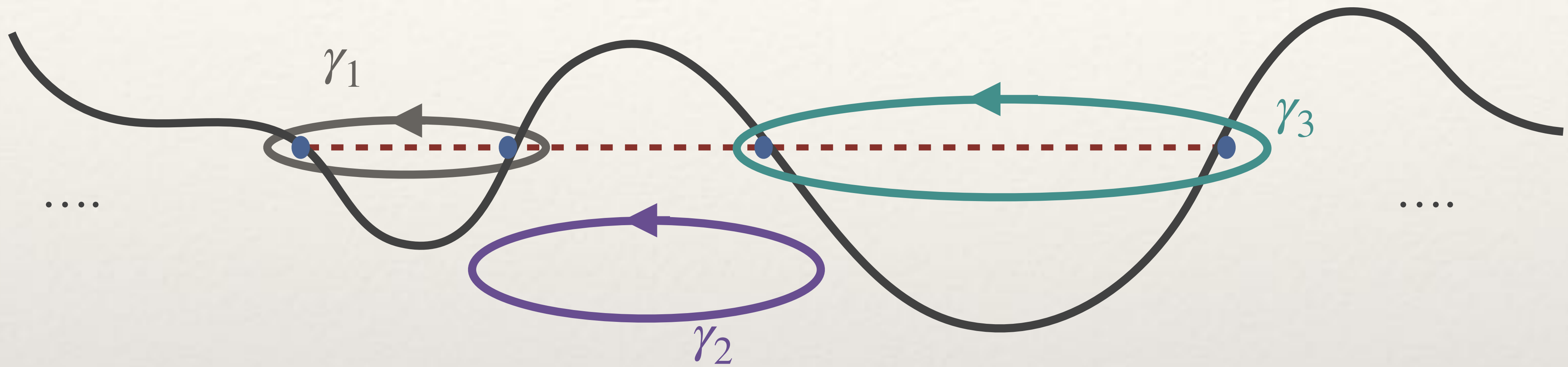
$$\langle \gamma_2, \gamma_1 \rangle = - \langle \gamma_2, \gamma_3 \rangle = -1$$

Intersection number between γ_1 and γ_2

$$\mathfrak{S}\mathscr{V}_{\gamma_1} = \mathscr{V}_{\gamma_1}(1 + \mathscr{V}_{\gamma_2})^{-1} \qquad \mathfrak{S}\mathscr{V}_{\gamma_2} = \mathscr{V}_{\gamma_2}$$

Delabaere Dillinger Pham formula

In general



$$\mathscr{V}_{\gamma_i} := e^{\frac{1}{\hbar}\oint_{\gamma_i} P(x;\hbar)dx}, \quad \gamma_i \in H_1(\Sigma)$$

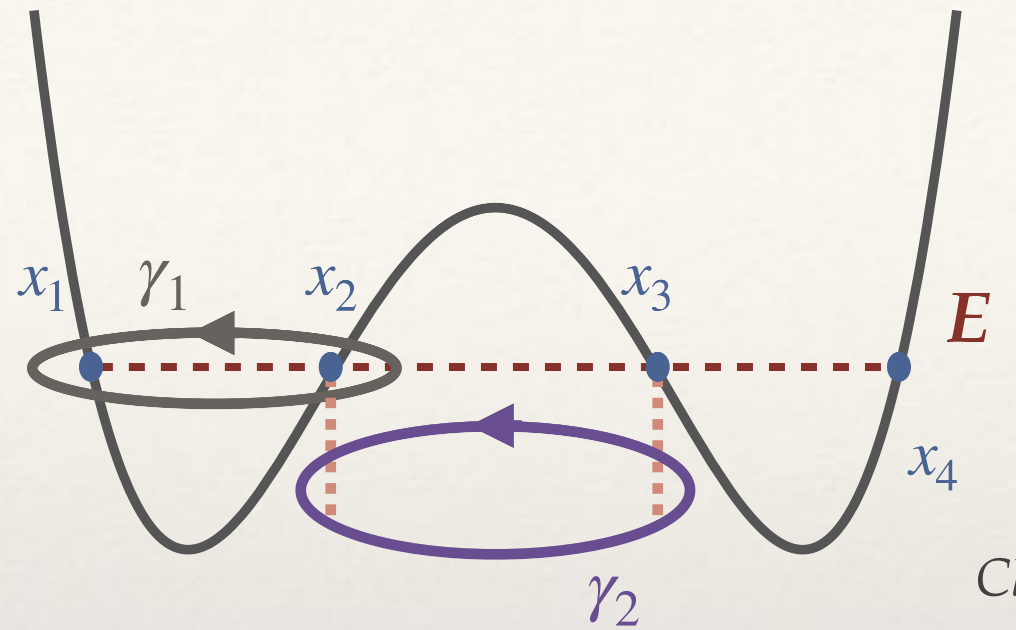
$$\mathfrak{S}\mathscr{V}_{\gamma_i} = \prod_j \mathscr{V}_{\gamma_i}(1+\mathscr{V}_{\gamma_j})^{\langle \gamma_j, \gamma_i \rangle}$$

for more complicate explicit examples see e.g.
[Sueishi, Kamata, Misumi, Ünsal, '20]

Double well: exact quantization

Exact quantization condition:

$$\mathscr{V}_{\gamma_1} - i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$$



Pure imaginary

$$\mathscr{V}_{\gamma_1} = e^{\frac{2}{\hbar}\int_{x_1}^{x_2} P(x;\hbar)dx} := e^{\frac{i}{\hbar}S_{pt}(E;\hbar)}$$

$$\mathscr{V}_{\gamma_2}^{1/2} = e^{-\frac{1}{\hbar}\int_{x_2}^{x_3} P(x;\hbar)dx} := e^{-\frac{1}{2\hbar}\hat{S}_{inst}(E;\hbar)}$$

Real

Classical action

Quantum corrections

$$S_{pt}(E) = 2\int_{x_2}^{x_1} \sqrt{2(E - V(x))}dx + \mathcal{O}(\hbar^2)$$

$$S_{inst}(E) = 2\int_{x_2}^{x_3} \sqrt{2(V(x) - E)}dx + \mathcal{O}(\hbar^2)$$

Tunneling ("instanton") action

Quantum corrections

Double well: perturbative quantization

If we neglect the exponentially small part: $\mathscr{V}_{\gamma_1} + 1 = 0$

$$S_{pt}(E) = 2\int_{x_2}^{x_1} \sqrt{2(E - V(x))}dx + \mathcal{O}(\hbar^2) = 2\pi\left(N + \frac{1}{2}\right)$$ *Bohr- Sommerfeld quantization*

$$\Rightarrow E_{pt}(N,\hbar) \approx \hbar\left(N + \frac{1}{2}\right) - \frac{\hbar^2}{4}\left(3\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right) + \dots$$

resurgent series

(perturbative expansion around the harmonic minimum, E=0)

Notice that there is no dependence on η. Perturbative spectrum is doubly degenerate.

Double well: perturbative quantization

$$S_{pt}(E) = \sqrt{2}\left(\oint_{\gamma_1}\sqrt{E-V}dx - \frac{\hbar^2}{2^6}\oint_{\gamma_1}\frac{(V')^2}{(E-V)^{5/2}}dx - \frac{\hbar^4}{2^{13}}\oint_{\gamma_1}\left(\frac{49(V')^4}{(E-V)^{11/2}} - \frac{16V'V'''}{(E-V)^{7/2}}\right)dx - \ldots\right)$$

$S_{pt,0}(E)$ $S_{pt,2}(E)$ $S_{pt,4}(E)$

$$S_{pt,0}(E) = 2\pi E\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},2;8E\right)$$

exercise: derive these expressions

$$S_{pt,2}(E) = \frac{1-12E}{16E(8E-1)}S_{pt,0}(E) + \frac{1}{8}\frac{dS_{pt,0}}{dE}$$

$$S_{pt,4}(E) = \frac{(-28+663E-5370E^2+11280E^3)}{7680E^3(-1+8E)^3}S_{pt,0}(E) + \frac{-7+115E-360E^2}{1920(1-8E)^2E^2}\frac{dS_{pt,0}(E)}{dE}$$

Expand around E=0 and invert the series $\hat{S}(E) = 2\pi\hbar(N+1/2)$ $(\hat{S}_0(E) = 2\pi E + 3\pi/2E^2 + \ldots)$

$$\Rightarrow E_{pt}(N,\hbar) \approx \hbar\left(N+\frac{1}{2}\right) - \frac{\hbar^2}{4}\left(3\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right) - \hbar^3\left(\frac{17}{16}\left(N+\frac{1}{2}\right)^3 + \frac{19}{64}\left(N+\frac{1}{2}\right)\right) + \ldots$$

Double well: exact quantization

Non-perturbatively: $\mathscr{V}_{\gamma_1} - i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$

$$\mathscr{V}_{\gamma_2}^{1/2} = e^{-\frac{1}{\hbar}\int_{x_2}^{x_3} P(x;\hbar)dx} := e^{-\frac{1}{2\hbar}S_{inst}(E;\hbar)} \approx c\, e^{-\frac{S_{inst}}{\hbar}}, \quad S_{inst} = \frac{2}{3}, \qquad \mathscr{V}_{\gamma_1} \approx e^{\frac{2\pi i}{\hbar}E}$$

(leading behavior around the harmonic minimum, E=0)

Ansatz: $E := \hbar\left(N + \frac{1}{2} + \epsilon\right) \Rightarrow e^{2\pi i(N+1/2+\epsilon)} + 1 - i\eta c e^{-\frac{S_{inst}}{\hbar}} = 0$

$$\epsilon = -2\pi c \eta e^{-\frac{S_{inst}}{\hbar}}$$

$$E(N;\hbar) = E_{pt}(N;\hbar) \pm 2\pi c e^{-\frac{S_{inst}}{\hbar}} + \dots$$

$$E_{odd} - E_{even} \sim e^{-\frac{S_{inst}}{\hbar}}$$

Non-perturbative level splitting

Double well: non-perturbative fluctuations

$$S_{inst}(E) = \sqrt{2}\left(\oint_{\gamma_2}\sqrt{E-V}dx - \frac{\hbar^2}{2^6}\oint_{\gamma_2}\frac{(V')^2}{(E-V)^{5/2}}dx - \frac{\hbar^4}{2^{13}}\oint_{\gamma_2}\left(\frac{49(V')^4}{(E-V)^{11/2}} - \frac{16V'V'''}{(E-V)^{7/2}}\right)dx - \dots\right)$$

$S_{inst,0}(E)$ $S_{inst,2}(E)$ $S_{inst,4}(E)$

$$S_{inst,0}(E) = \frac{\pi}{2\sqrt{2}}(1-8E)\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},2;1-8E\right)$$

$$S_{inst,2}(E) = \frac{1-12E}{16E(8E-1)}S_{inst,0}(E) + \frac{1}{8}\frac{S_{inst,0}(E)}{dE}$$

$$\vdots$$

$$\mathcal{V}_{\gamma_2}^{1/2} \sim e^{-\frac{S_{inst}}{\hbar}} I_{fluc}(E;\hbar)$$

Instanton fluctuations

$$e^{\frac{i}{\hbar}S_{pt}(E;\hbar)} - i\eta e^{-\frac{S_{inst}}{\hbar}} I_{fluc}(E;\hbar) + 1 = 0$$

See also [Zinn-Justin, Jentschura; Dunne, Unsal…]

End of Lecture II