WKB, Eigenvalue Problems and Quantisation in QM

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Spring school on asymptotic methods and applications

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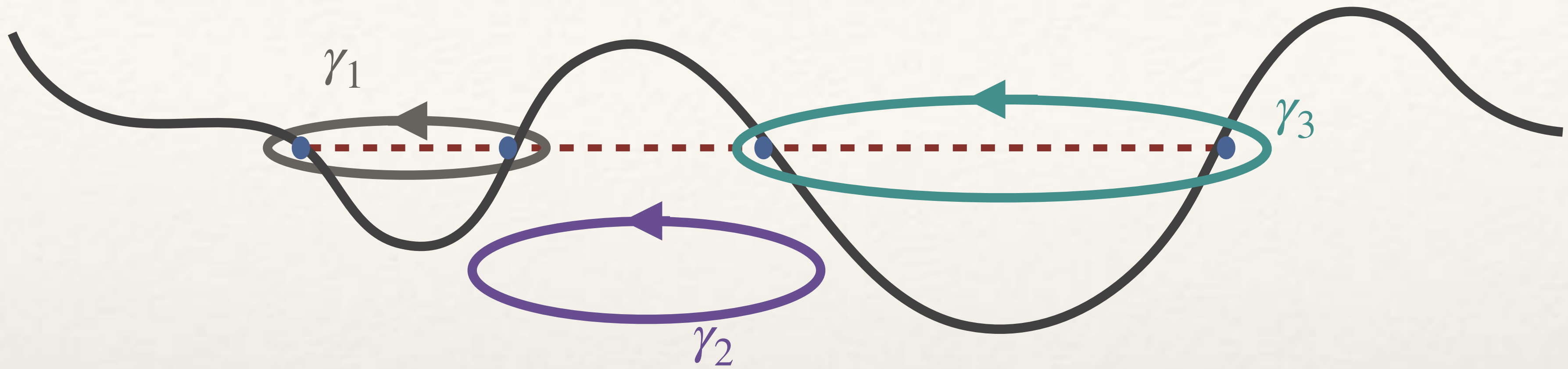
Recap of lecture II



- Local WKB solutions are glued together via *Voros symbols*

$$\mathscr{V}_{\gamma_i} := e^{\frac{1}{\hbar}\oint_{\gamma_i} P(x;\hbar)dx}, \quad \gamma_i \in H_1(\Sigma)$$

- Voros symbols are resurgent functions and their Stokes automorphisms are given by the *Delebaere Dilinger Pham formula*:

$$\mathfrak{S}\mathscr{V}_{\gamma_i} = \prod_j \mathscr{V}_{\gamma_i}(1 + \mathscr{V}_{\gamma_j})^{\langle \gamma_j, \gamma_i \rangle}$$

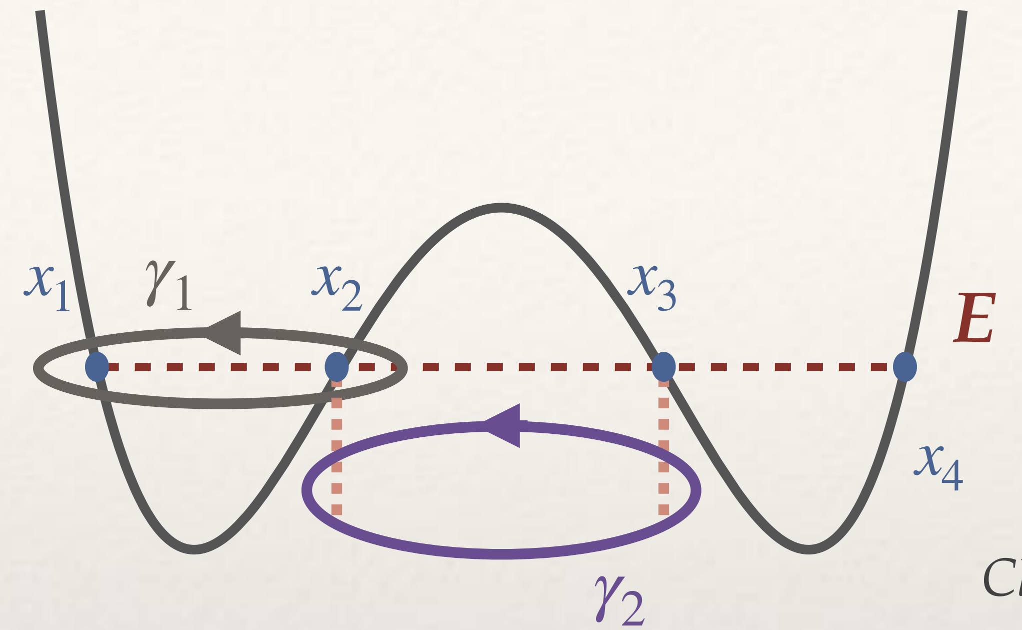
- The gluing can only be done (for well-behaved wave-functions) for certain values of E : *Voros Silverstone connection formula*: $f(\mathscr{V}_{\gamma_1}, \mathscr{V}_{\gamma_2}, \ldots) = 0$

see e.g [Ito, Marino, Shu '18; Hollands, Nietzke '19] for a derivation via Wronskians

Double well, continued...

Exact quantization condition:

$$\mathscr{V}_{\gamma_1} - i\eta \mathscr{V}_{\gamma_2}^{1/2} + 1 = 0$$



Pure imaginary

$$\mathscr{V}_{\gamma_1} = e^{\frac{2}{\hbar}\int_{x_1}^{x_2} P(x;\hbar)dx} := e^{\frac{i}{\hbar}S_{pt}(E;\hbar)}$$

$$\mathscr{V}_{\gamma_2}^{1/2} = e^{-\frac{1}{\hbar}\int_{x_2}^{x_3} P(x;\hbar)dx} := e^{-\frac{1}{2\hbar}S_I(E;\hbar)}$$

Real

Classical action

Quantum corrections

$$S_{pt}(E) = 2\int_{x_2}^{x_1} \sqrt{2(E - V(x))}dx + \mathcal{O}(\hbar^2)$$

$$S_I(E) = 2\int_{x_2}^{x_3} \sqrt{2(V(x) - E)}dx + \mathcal{O}(\hbar^2)$$

Tunneling ("instanton") action

Quantum corrections

Double well: non-perturbative fluctuations

$$e^{\frac{i}{\hbar}S_{pt}(E;\hbar)} - i\eta e^{-\frac{S_I}{\hbar}}F_{inst}(E;\hbar) + 1 = 0 \qquad (S_I = 2/3)$$

Leading order ansatz $E := \hbar\left(N + \frac{1}{2} + \epsilon\right)$ $\qquad e^{\frac{-1}{2\hbar}S_I(E;\hbar)} := e^{-\frac{S_I}{\hbar}}F_{inst}(E;\hbar)$

- $S_{pt}(E;\hbar) \approx 2\pi E$
- $S_I(E;\hbar) \approx \frac{4}{3} + 2E\left(\log\left(\frac{E}{8}\right) - 1\right) + \ldots \Rightarrow \qquad F_{inst}(E;\hbar) \sim \hbar^{N+1/2+\epsilon} \sim \hbar^{N+1/2}(1 + \epsilon\log\hbar + \ldots)$

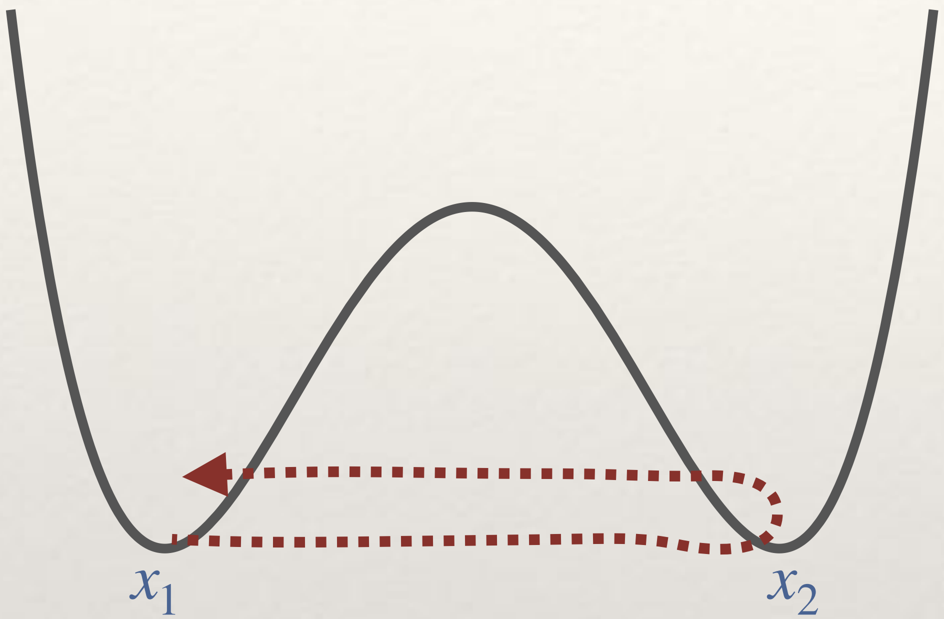
$$\Rightarrow \epsilon \sim c_I e^{-\frac{S_I}{\hbar}} + c_{I\bar{I}}e^{-2\frac{S_I}{\hbar}}(\log\hbar + \ldots)$$

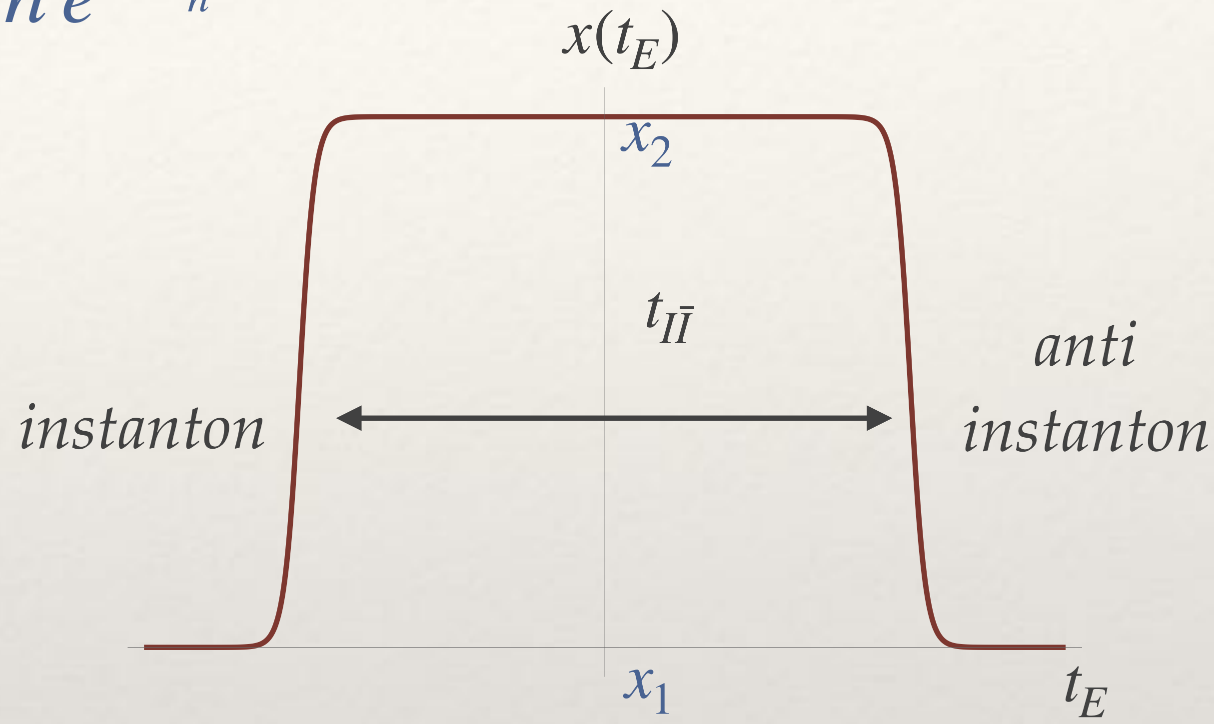
log terms show up in the 2-instanton (anti - instanton) sector

$$E_{I\bar{I}} \sim \log\hbar\, e^{-\frac{2S_{inst}}{\hbar}}$$

Double well: instanton-anti-instanton

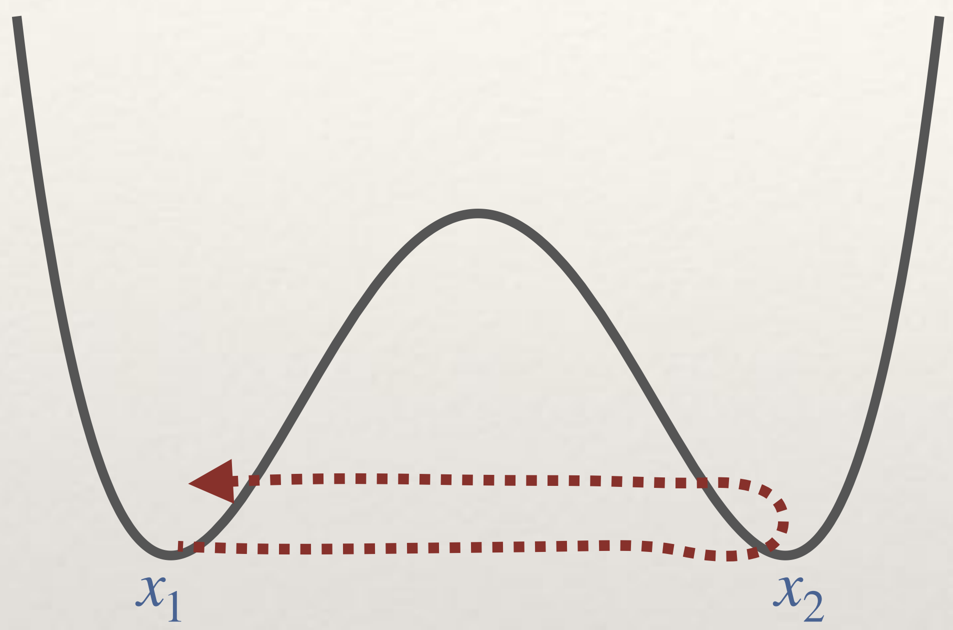
$$E_{I\bar{I}} \sim \log \hbar \, e^{-\frac{2S_{inst}}{\hbar}}$$

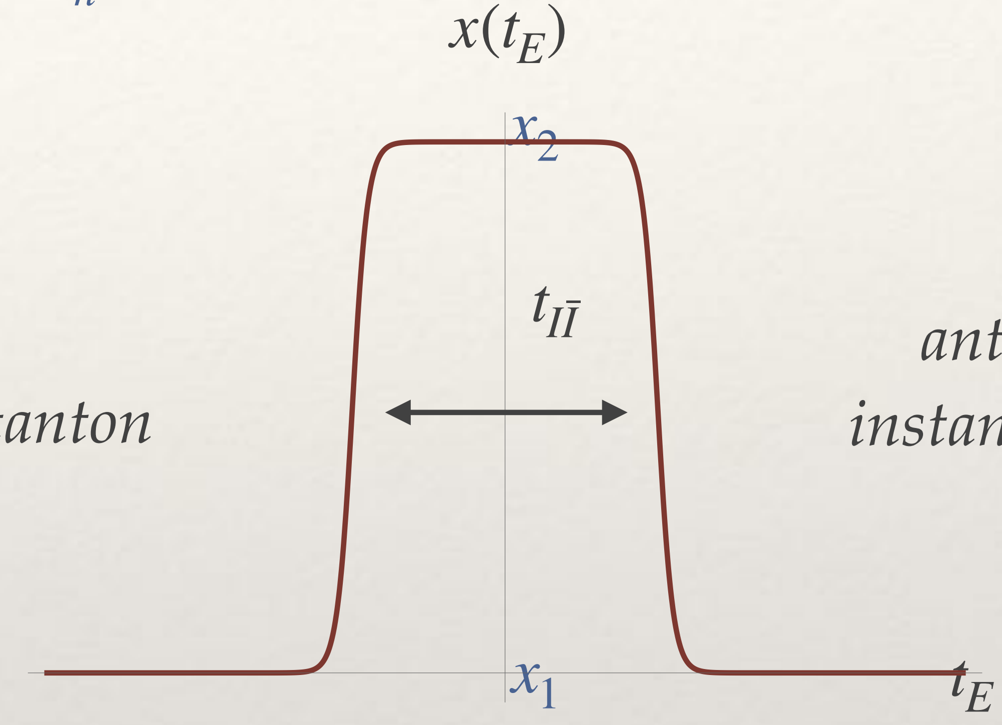




Double well: instanton-anti-instanton

$$E_{I\bar{I}} \sim \log \hbar \, e^{-\frac{2S_{inst}}{\hbar}}$$





- $t_{I\bar{I}}$:"quasi zero mode"
- Summing over $t_{I\bar{I}}$ in the path integral $\Rightarrow \log \hbar \, e^{-\frac{2S_{inst}}{\hbar}}$

See for e.g. [Dunne, Ünsal; Uniform WKB]

Double well: non-perturbative fluctuations

$$e^{\frac{i}{\hbar}S_{pt}(E;\hbar)} - i\eta e^{-\frac{S_{inst}}{\hbar}} I_{fluc}(E;\hbar) + 1 = 0$$

perturbative series (via Bohr-Sommerfeld) diverge as

$$c_{0,n} \sim \frac{n!}{(S_{inst})^n}(c_{1,0} + \ldots) + \frac{n!}{(2S_{inst})^n}(c_{2,0} + \ldots)$$

1-instanton fluctuations

$$c_{1,n} \sim \frac{n!}{(S_{inst})^n}(c_{2,0} + \ldots) + \frac{n!}{(2S_{inst})^n}(c_{3,0} + \ldots)$$

$$E(N,\hbar) \sim \sum_{n=0}^{\infty} c_{0,n}(N+1/2)\hbar^n + \eta e^{-\frac{S_{inst}}{\hbar}} \sum_{n=0}^{\infty} c_{1,n}(N+1/2)\hbar^n$$

$$+ \sum_{l=2}^{\infty}\sum_{k=1}^{l-1}\sum_{n=0}^{\infty} \eta^l c_{l,k,n}(N+1/2)\hbar^n (\log\hbar)^k e^{-l\frac{S_{inst}}{\hbar}} + \ldots$$

multi instantons

Double well, continued...

$$e^{\frac{1}{\hbar}S_{\gamma_1}(E;\hbar)} - i\eta e^{-\frac{1}{2\hbar}S_{\gamma_2}(E;\hbar)} + 1 = 0$$

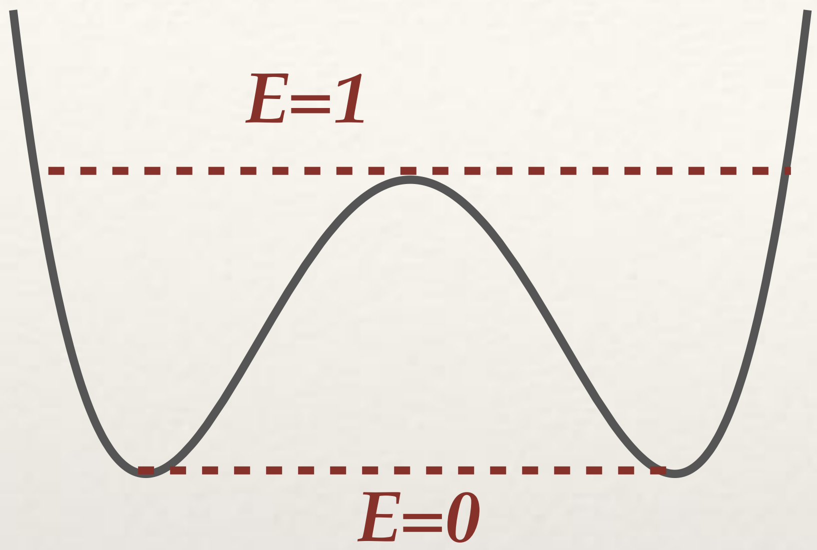
$$S_{\gamma_i} = \oint_{\gamma_i} \left(P_0(x) + \hbar^2 P_2(x) + \ldots\right) = S_{\gamma_i,0}(E) + \hbar^2 S_{\gamma_i,2}(E) + \ldots$$

Recall $$S_{\gamma_i,n}(E) = P_n(E) S_{\gamma_i,0} + Q_n(E) \frac{S_{\gamma_i,0}}{dE}$$

Rational functions of E independent of γ_i !

- All the higher order actions are related to the classical action in a simple way!

Double well, classical limit



$V(x) = (2x^2 - 1)^2$

Periods of the torus

$$\omega_{\gamma_1,0}(E) = \frac{dS_{\gamma_1,0}(E)}{dE} = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},1;E\right)$$

$$\omega_{\gamma_2,0}(E) = \frac{dS_{\gamma_2,0}(E)}{dE} = i\frac{\pi}{\sqrt{2}}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},1;1-E\right)$$

$$\tau = \frac{\omega_{\gamma_2,0}}{2\omega_{\gamma_1,0}}$$

modular parameter

Picard-Fuchs equation

Two independent solutions of $E(1-E)\Omega''(E) + (1-2E)\Omega'(E) - \frac{3}{16}\Omega(E) = 0$

Double well, classical limit

Two independents solutions of $E(1-E)\Omega''(E)+(1-2E)\Omega'(E)-\frac{3}{16}\Omega(E)=0$

- The periods, in general (for genus 1 spectral curves), satisfy a 3rd order ODE, but for double well case remarkably (*will generalize soon…*):

$$E(1-E)S_0''(E)-\frac{3}{16}S_0(E)=0$$ (*will call this Picard Fuchs eqn as well*)

- $S_{\gamma_1,0}(E),\ S_{\gamma_2,0}(E)$ are two independent solutions

Wronskian: $W[S_{\gamma_1,0},S_{\gamma_2,0}]=S_{\gamma_1,0}(E)\omega_{\gamma_2,0}(E)-S_{\gamma_2,0}(E)\omega_{\gamma_1,0}(E)=2iS_{inst}T$

Riemann bilinear identity

see e.g. [Gorsky, Milekhin]

Matone relation (SW theory)

$S_{\gamma_2,0}(0)\omega_{\gamma_1,0}(0)=-2iS_{inst}T$

T= period at the bottom of the well

Quantum actions, general genus-1

- Each $S_{\gamma_i,n}(E)$ is an elliptic integral that take value in $H_1(T^2)$.
- The integrands are generated recursively from $\sqrt{Q(x)}$ no additional singularities
- They must be a combination of three independent elliptic integrals $\mathbb{K}, \mathbb{E}, \Pi$ [Weierstrass]
- They are closed under differentiation

n^{th} order diff. operator

$$S_{\gamma_i,n}(E) = \mathscr{D}^n S_{\gamma_i,0}(E)$$

Using the Picard-Fuchs equation for S_0 *(in general 3^{rd} order)*

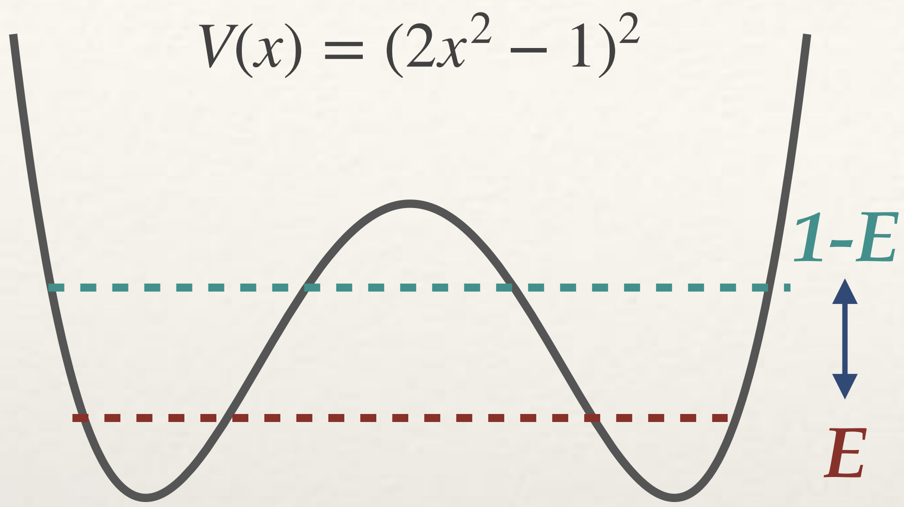
$$S_{\gamma_i,n}(E) = f_n(E) S_{\gamma_i,0}(E) + g_n(E) S'_{\gamma_i,0}(E) + h_n(E) S''_{\gamma_i,0}(E)$$

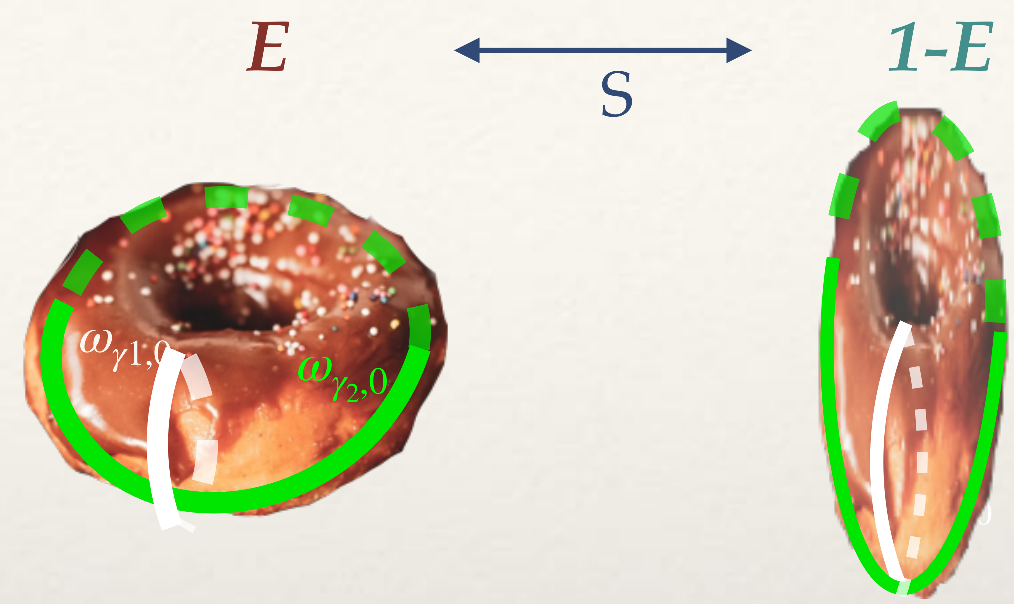
True for any genus 1 potential!

- *f, g, h:* fairly easy to generate
- For double well PF is 2^{nd} order $S_{\gamma_i,n}(E) = f_n(E) S_{\gamma_i,0}(E) + g_n(E) S'_{\gamma_i,0}(E)$

[GB, Dunne, Ünsal]

Double well, modular symmetry





$$\omega_{\gamma_1,0}(E) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},1;E\right)$$

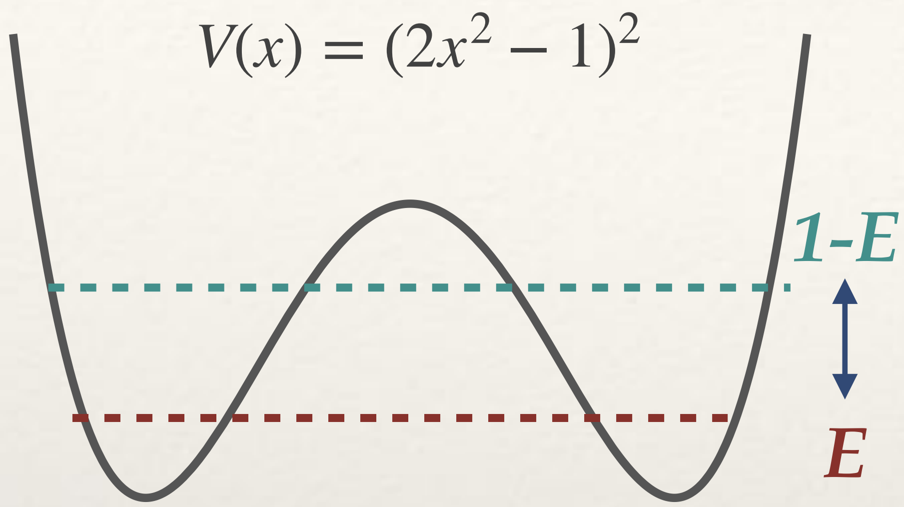
$$\tau(E) = \frac{\omega_{\gamma_2,0}(E)}{2\omega_{\gamma_1,0}(E)},\quad \tau(1-E) = \frac{\omega_{\gamma_2,0}(1-E)}{\omega_{\gamma_1,0}(1-E)} = -\frac{1}{2\tau(E)}$$

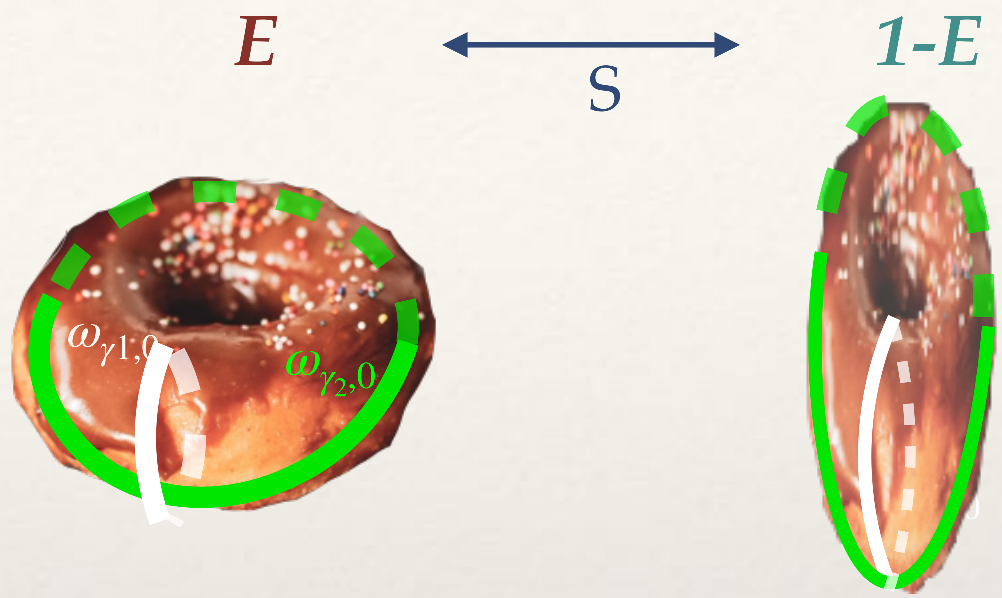
$$\omega_{\gamma_2,0}(E) = i\frac{\pi}{\sqrt{2}}\ {}_2F_1\left(\frac{1}{4},\frac{3}{4},1;1-E\right)$$

Modular transformations:

$$\text{S: } \tau \to -\frac{1}{2\tau},\quad \text{T: } \tau \to \tau + 1$$

Double well, modular symmetry





Use τ as the modular parameter (instead of E)

modular form

S: $\omega_{\gamma 1,0}(-1/(2\tau)) = -i\sqrt{2}\tau\omega_{\gamma 1,0}(\tau)$

T: $\omega_{\gamma 1,0}(\tau+1) = \omega_{\gamma 1,0}(\tau)$

S: $S_{\gamma 2,0}(\tau) = S_{\gamma 1,0}(-1/(2\tau)) = 2\tau S_{\gamma_1,0} + i\dfrac{S_{inst}T}{2\omega_{\gamma_2,0}}$

Bilinear (Wronskian, Matone,…) identity!

quasi-modular form

Double well, quantum actions

what happens to the *bilinear identity* (Wronskian)?

$$S_{\gamma_1,0}(E)\omega_{\gamma_2,0}(E) - S_{\gamma_2,0}(E)\omega_{\gamma_1,0}(E) = -2S_{inst}T$$

Double well, P/NP relation

what happens to the *bilinear identity* (Wronskian)?

$$\left(S_{\gamma_1}(E;\hbar) - \hbar \frac{S_{\gamma_1}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_2}(E;\hbar) - \left(S_{\gamma_2}(E;\hbar) - \hbar \frac{S_{\gamma_2}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_1}(E;\hbar) = 2iS_{inst}T$$

Only new terms!

Double well, P/NP relation

what happens to the *bilinear identity* (Wronskian)?

$$\left(S_{\gamma_1}(E;\hbar) - \hbar \frac{S_{\gamma_1}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_2}(E;\hbar) - \left(S_{\gamma_2}(E;\hbar) - \hbar \frac{S_{\gamma_2}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_1}(E;\hbar) = 2iS_{inst}T$$

Only new terms!

[Álvarez, Cesares; Dunne, Ünsal, …]

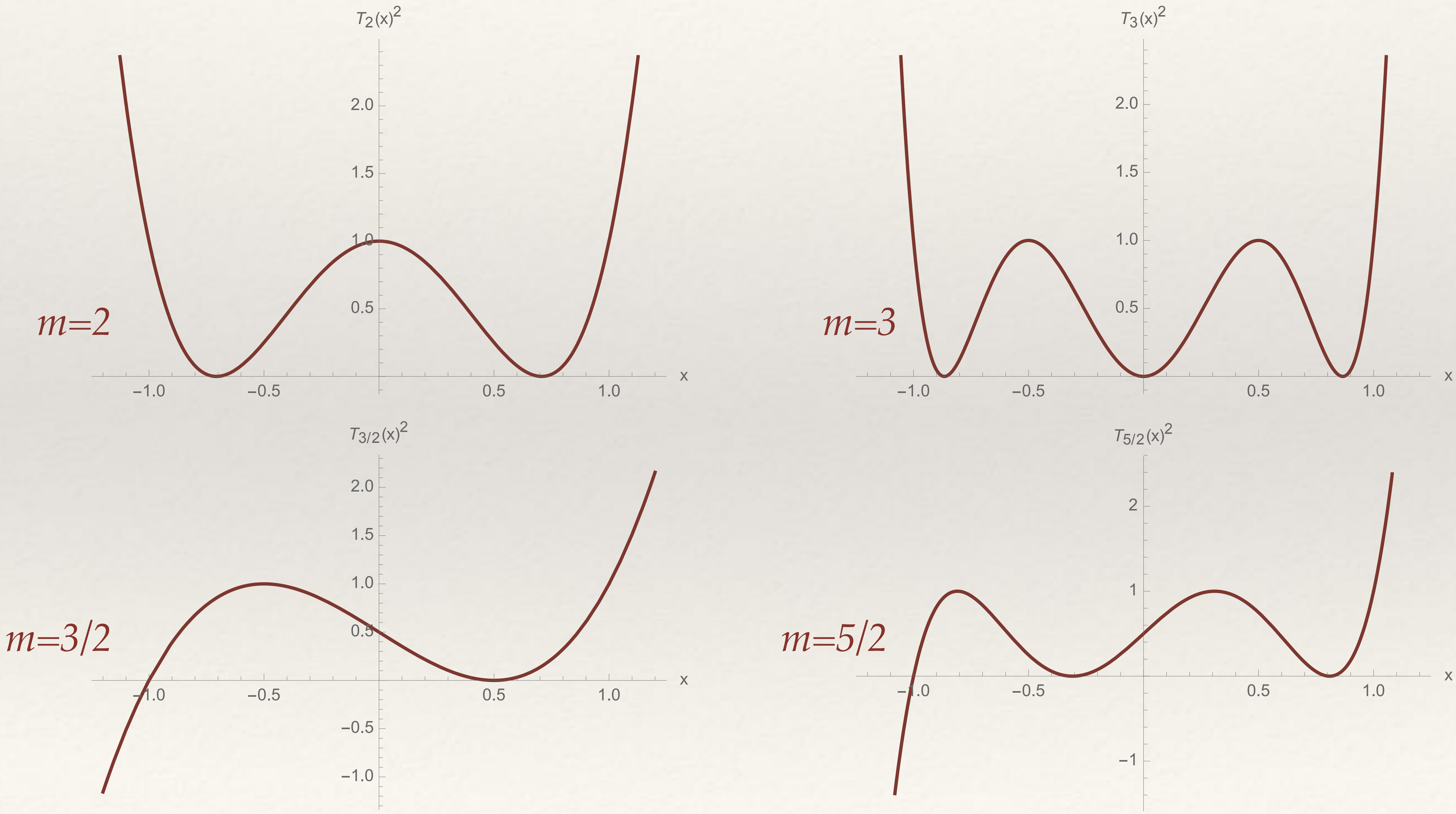
- Exact (all orders in $\hbar$) relation
- Valid everywhere in the spectrum
- Relates perturbative and non-perturbative expansions order by order *("P=NP")* (*in addition* to the resurgent large order/low order relations)
- How general is it?

see also e.g.
[Codesido,Marino, Schiappa '18] *holomorphic anomaly,*
[Gorsky, Milkehin, '14] *Whitham hierarchy*

P/NP relation and modularity

consider the class of potentials $V(x) = T_m^2(x)$

T_m: m^{th} Chebychev polynomial
m: half-integer, m wells

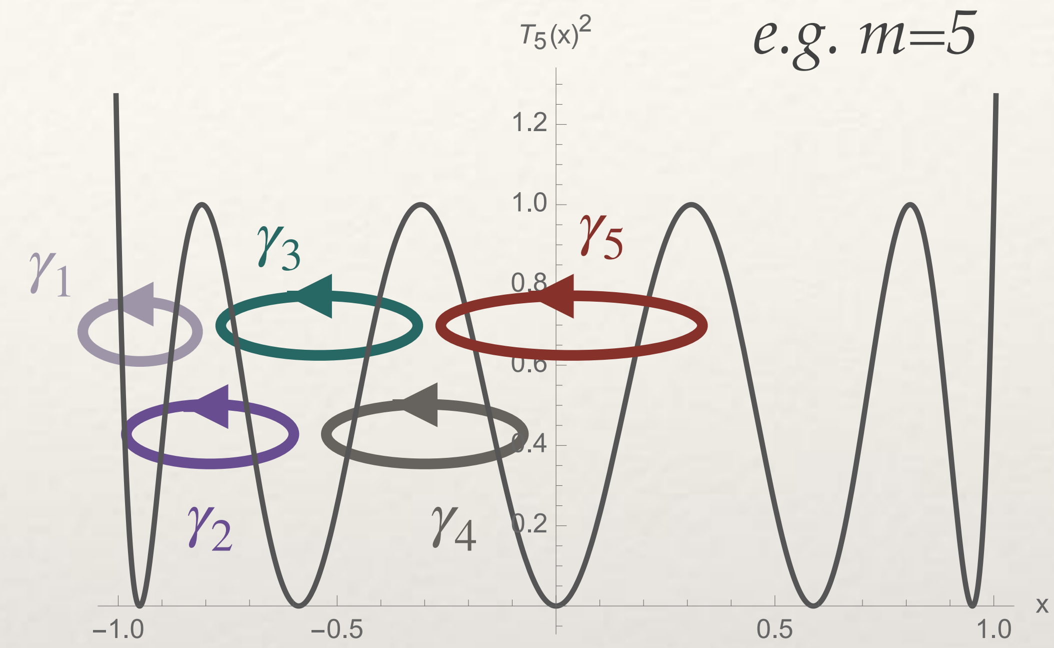


P/NP and classical geometry

- Only two independent periods!

$$\omega_{\gamma_1,0}(E) \propto \omega_{\gamma_1,0}(E) \propto \omega_{\gamma_5,0}(E) \propto \dots$$

$$\omega_{\gamma_2,0}(E) \propto \omega_{\gamma_4,0}(E) \propto \omega_{\gamma_6,0}(E) \propto \dots$$



$$\omega_{\gamma_1,0}(E) = \frac{\sqrt{2}\pi}{m} \sin\left(\frac{\pi}{2m}\right) \; {}_2F_1\left(\frac{1}{2} - \frac{1}{2m}, \frac{1}{2} + \frac{1}{2m}, 1; E\right)$$

$$\omega_{\gamma_2,0}(E) = i\frac{\sqrt{2}\pi}{m} \sin\left(\frac{\pi}{m}\right) \; {}_2F_1\left(\frac{1}{2} - \frac{1}{2m}, \frac{1}{2} + \frac{1}{2m}, 1; 1 - E\right)$$

$$\frac{4\pi i}{m^2 - 1} \sin^2\left(\frac{\pi}{m}\right)$$

$$||$$

$$E(1-E)S_0''(E) - \frac{1}{4}\left(1 - \frac{1}{m^2}\right) S_0(E) = 0$$

$$S_{\gamma_1,0}(E)\omega_{\gamma_2,0}(E) - S_{\gamma_2,0}(E)\omega_{\gamma_1,0}(E) = 2iS_{inst}T$$

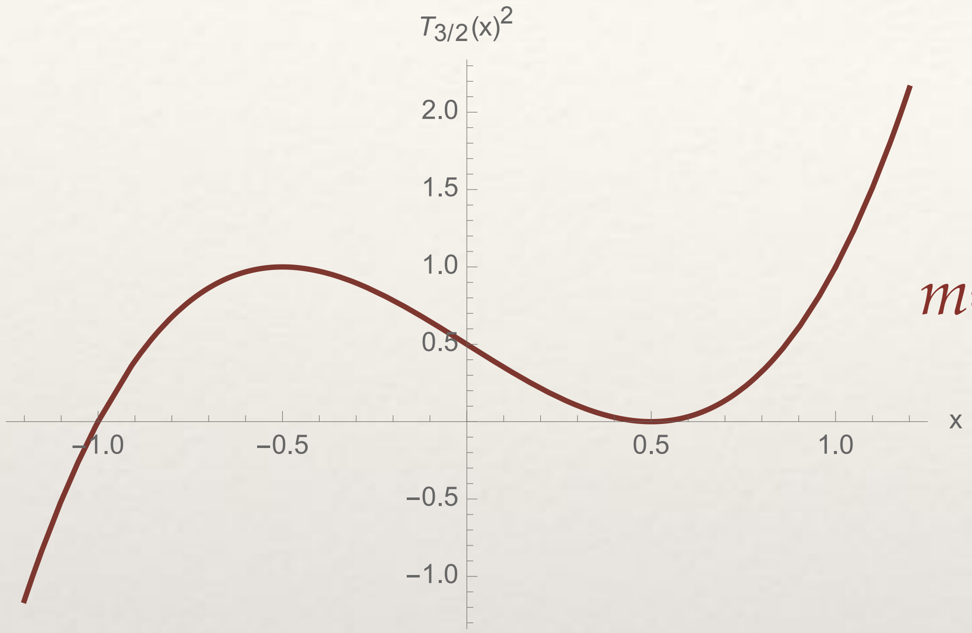
Picard-Fuchs

Bilinear identity

What happens after quantization..?

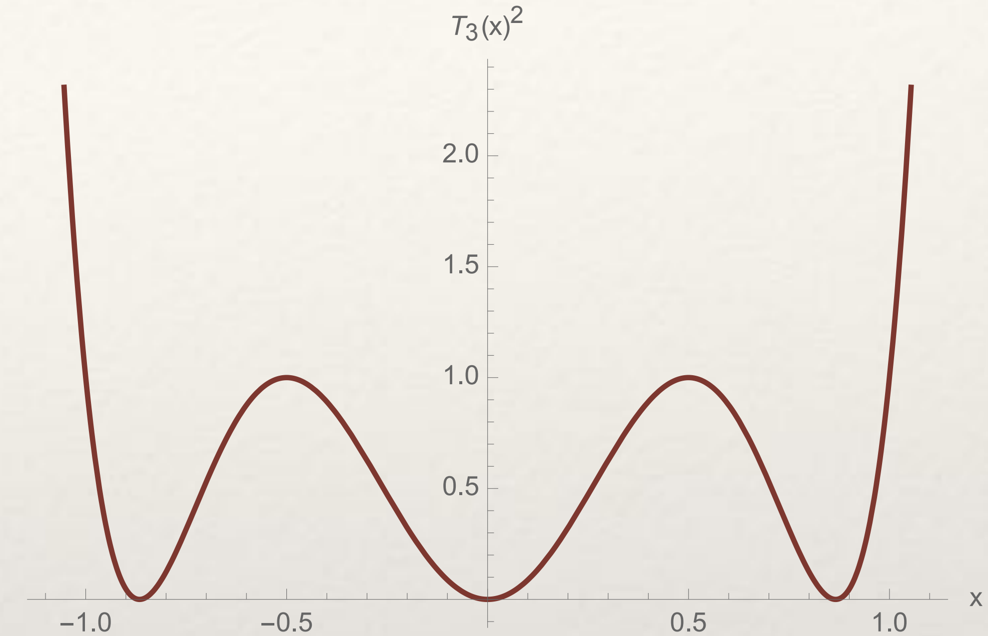
Quantum properties

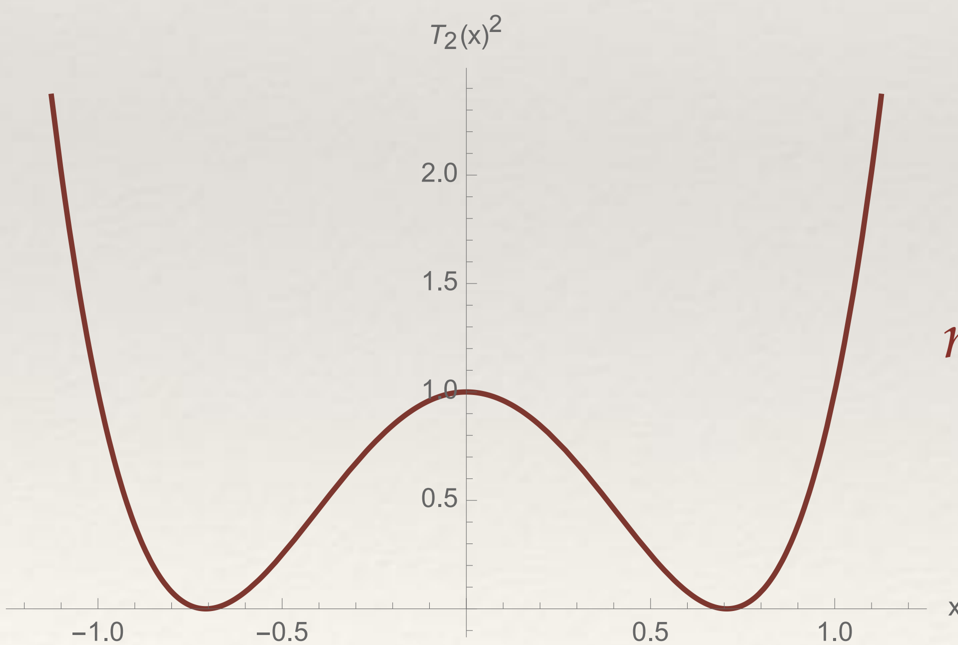
The quantum bilinear identity survives only for 4 cases *



m=3/2

m=3

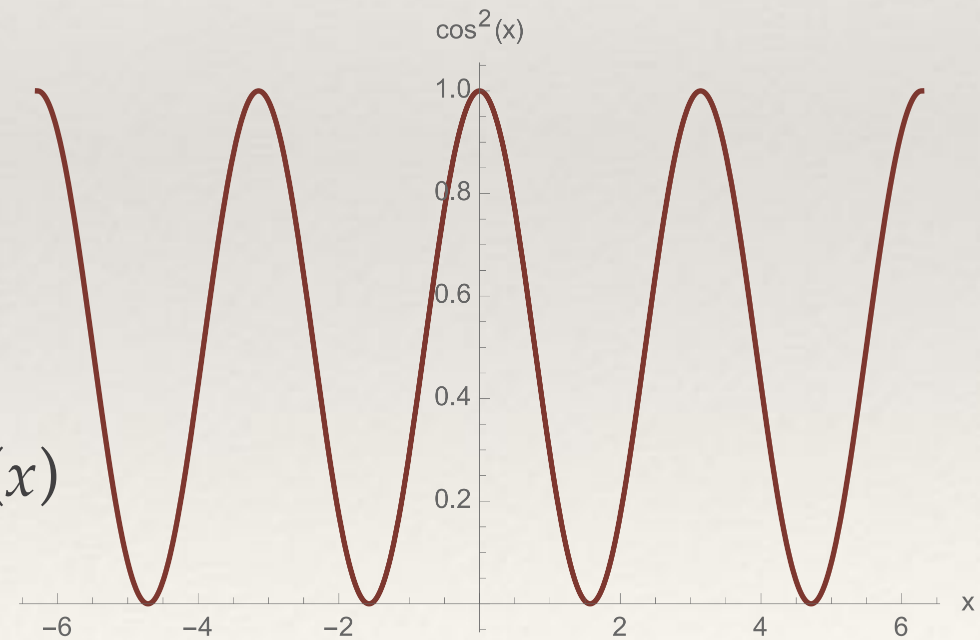




m=2

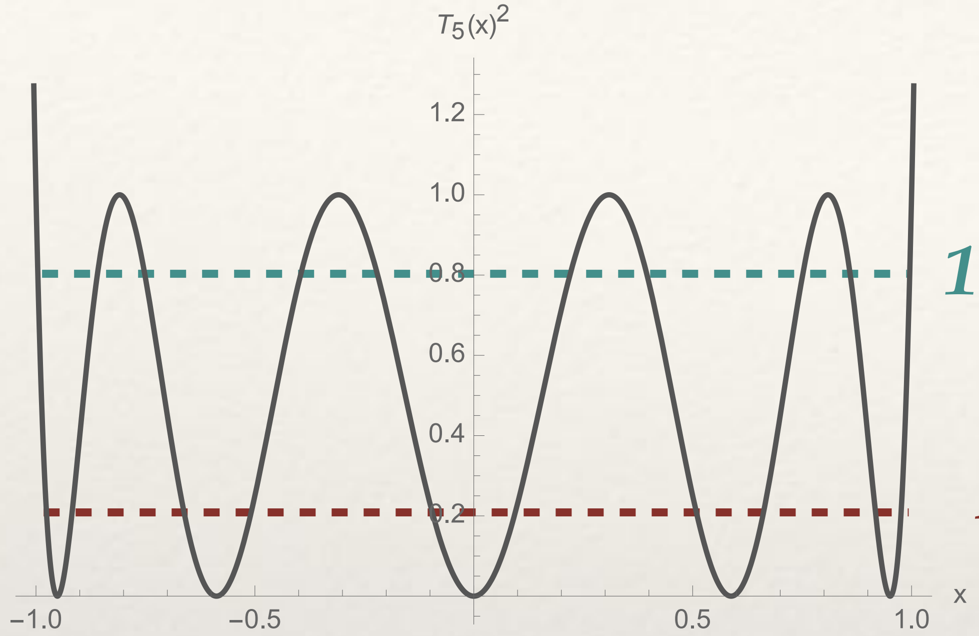
m=∞

$V(x)=\cos^2(x)$



*see [Raman, Bala Subramanian, '20] for a refinement of this statement

Modular vs Hecke



$$\text{S: } \tau \to -\frac{1}{r\tau}, \quad \text{T: } \tau \to \tau + 1 \qquad r = 4\cos^2\left(\frac{\pi}{2m}\right)$$

Hecke group

- Only when r=integer, *Hecke group* ~ $\Gamma_0(r)$ *(congruence subgroup of the modular group)*

- r=integer $\Leftrightarrow m = 3/2, 2, 3, \infty$

Quantum properties

$m = 3/2, 2, 3, \infty \Leftrightarrow$ Ramanujan's elliptic functions in alternative bases

$e^{i\pi\tau}$

$x \leftrightarrow E$

$m=\infty$

Example 1

$$\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)=\frac{x}{16}\left(1+\frac{1}{2}x+\frac{21}{64}x^2+\cdots\right).$$

$m=3$

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right)=\frac{x}{27}\left(1+\frac{5}{9}x+\cdots\right).$$

$m=2$

Example 3

$$\exp\left(-\sqrt{2}\pi\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right)=\frac{x}{64}\left(1+\frac{5}{8}x+\cdots\right).$$

$m=3/2$

Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right)=\frac{x}{432}\left(1+\frac{13}{18}x+\cdots\right).$$

[Berndt, Ramanujan's Notebooks Vol. II]

Quantum properties

$m = 3/2, 2, 3, \infty \Leftrightarrow$ Ramanujan's elliptic functions in alternative bases

$e^{i\pi\tau}$

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Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right)=\frac{x}{432}\left(1+\frac{13}{18}x+\cdots\right).$$

[Berndt, Ramanujan's Notebooks Vol. II]

We do not know Ramanujan's intention in giving Examples 1–4.

Quantum properties

$m = 3/2, 2, 3, \infty \Leftrightarrow$ Ramanujan's elliptic functions in alternative bases

$$P^2(x;\hbar) - \hbar\frac{dP}{dx} = Q(x)$$

$m{=}\infty$ $\quad - x)$ $\quad ; 1; x)$

$e^{i\pi\tau}$

$m{=}3$

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;}\right.$$

$x \leftrightarrow E$

$m{=}2$

Example 3

$$\exp\left(-\sqrt{2}\pi\,\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1}{{}_2F_1(\frac{1}{4},\frac{3}{4}}\right.$$

$m{=}3/2$

Example 4

$$\exp\left(-2\pi\,\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1}{{}_2F_1(\frac{1}{6},\frac{5}{6};1}\right.$$

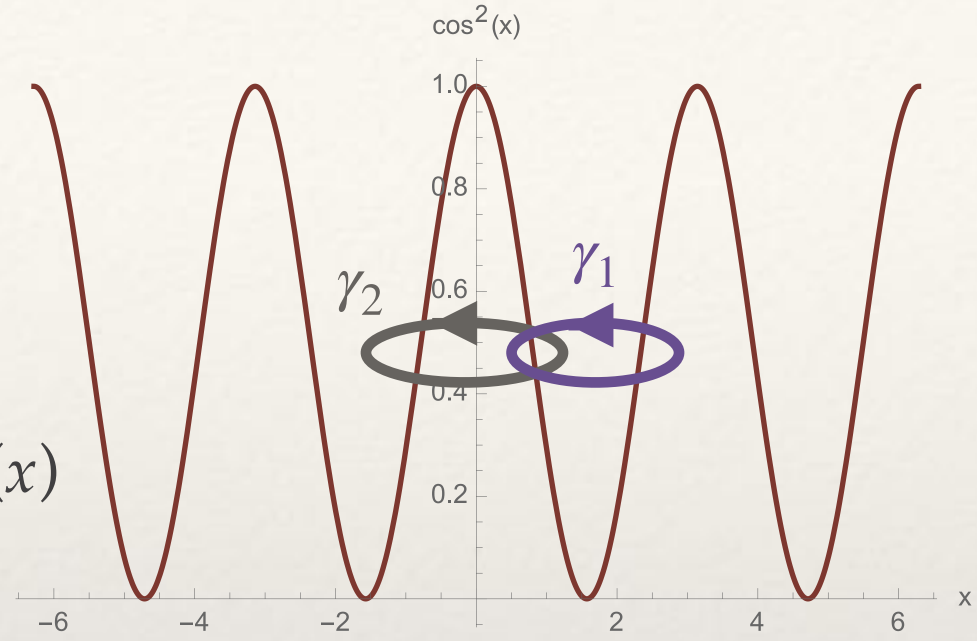
ks Vol. II]

We do not know Ramanujan's

Mathieu equation

m=∞

$V(x)$=cos²(x)



Bloch b.c. $\psi(x+\pi) = e^{i\theta}\psi(x)$

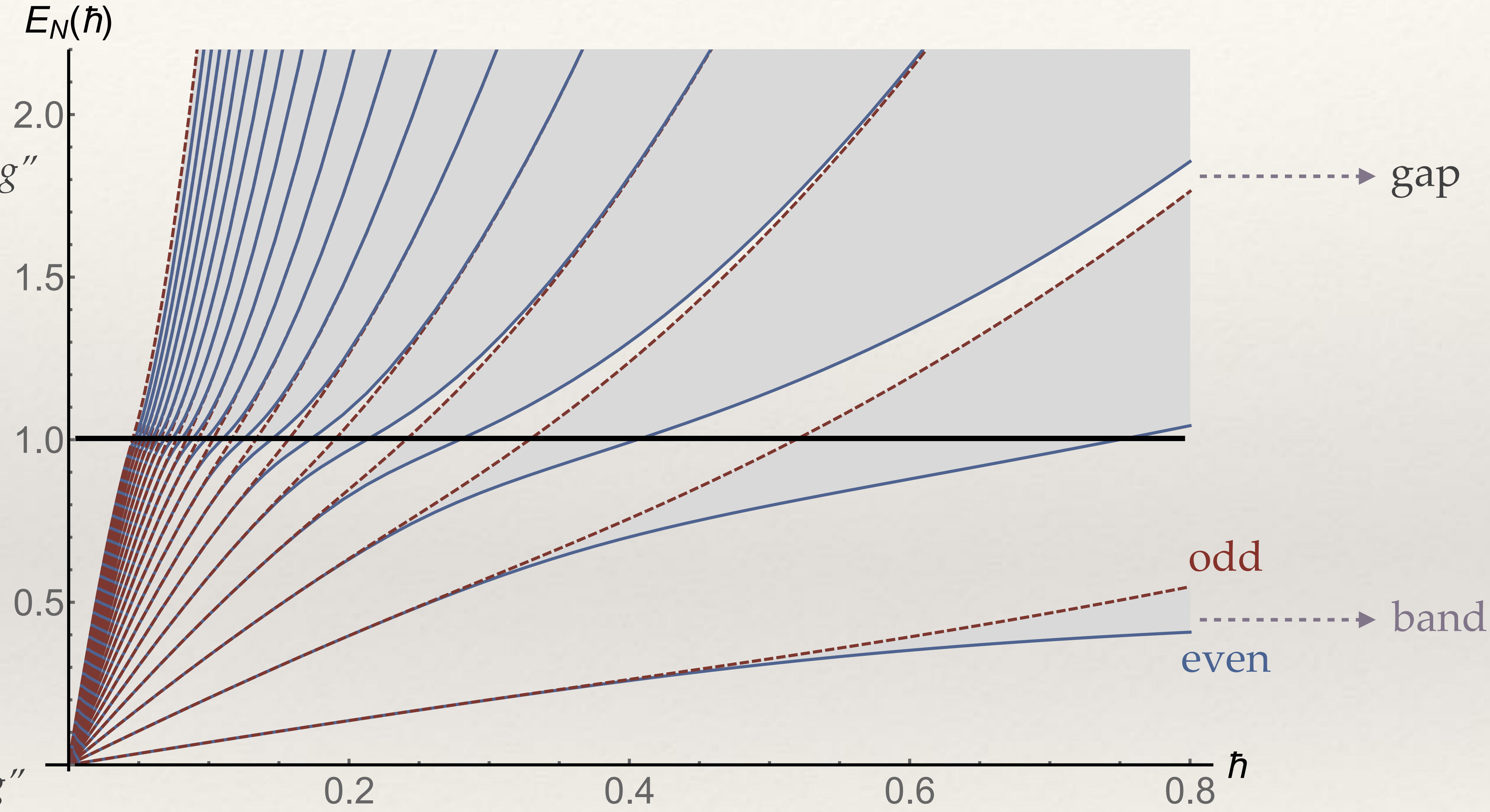
exercise: show that with an appropriate change of variables for p and x the spectral curve is a g=1 elliptic curve.

- Two independent cycles: γ_1, γ_2
- P/NP relation: $$\left(S_{\gamma_1}(E;\hbar) - \hbar \frac{S_{\gamma_1}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_2}(E;\hbar) - \left(S_{\gamma_2}(E;\hbar) - \hbar \frac{S_{\gamma_2}(E;\hbar)}{\partial \hbar} \right) \omega_{\gamma_1}(E;\hbar) = 8\pi i$$

Mathieu Spectrum

$N\hbar \gg 1$

"strong coupling"



$N\hbar \ll 1$

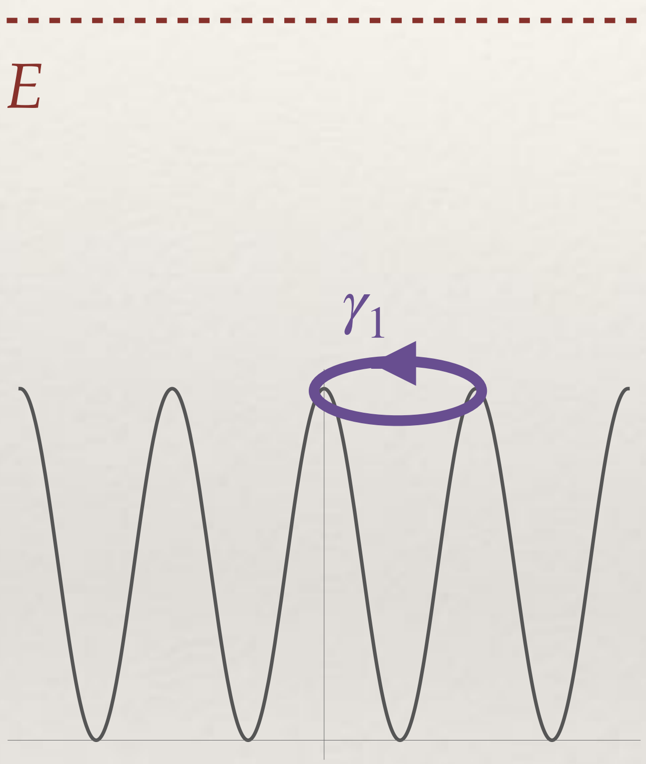
"weak coupling"

$$E(\hbar) \sim \sqrt{2}\hbar(N+1/2) - \left(\frac{1}{4}(N+1/2)^2 - \frac{1}{16}\right)\hbar^2 + \ldots - \cos\theta\, e^{-\frac{2\sqrt{2}}{\hbar}} I(N;\hbar) + \ldots$$

band width

Mathieu, strong coupling

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos^2(x)\psi = E\psi$$



- Leading order term: free particle $\psi = e^{i\nu x}$
- Perturbative corrections in $\hbar^{-4}$ (with $\hbar\nu \gg 1$)
- Covergent expansion!
- Bohr -Sommerfeld quantization: ν := N= integer

$$\nu = \frac{1}{2\pi}S_{\gamma_1} = 2\pi N$$

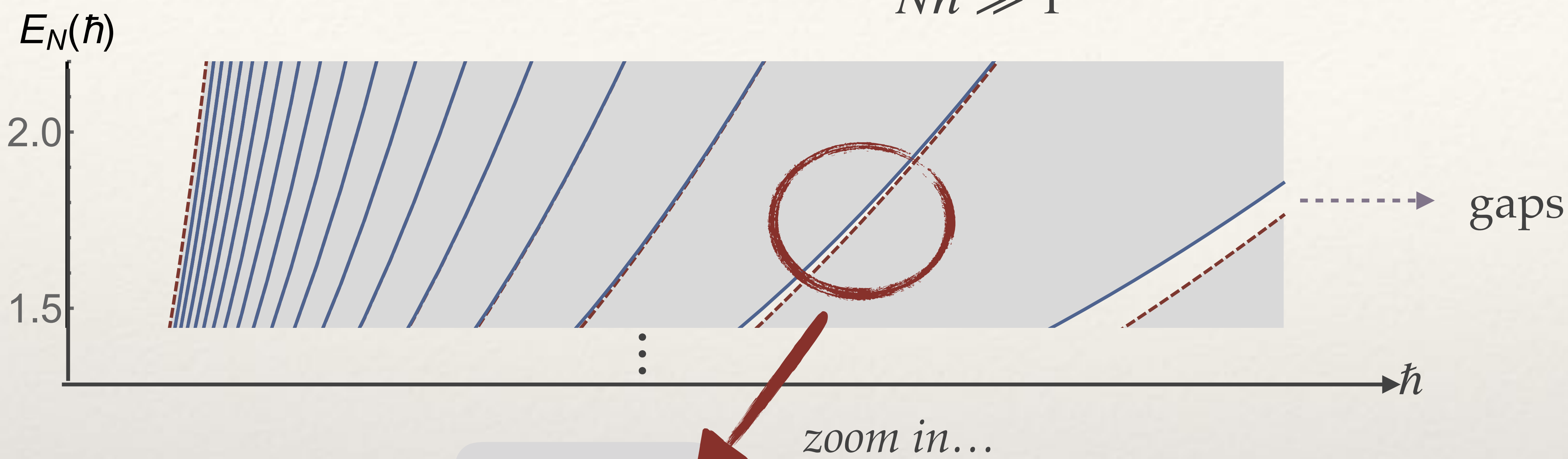
"type A quantization"
[Nekrasov, Shatashvili]

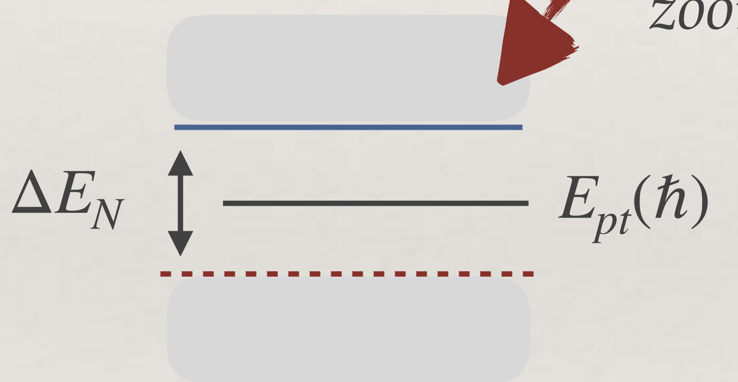
- Poles for even/odd states (ν := N= integer)

$$E_{pt}(\hbar) \sim \frac{\hbar^2}{2}\left(\nu^2 + \frac{1}{8(\nu^2-1)}\left(\frac{1}{\hbar}\right)^4 + \frac{5\nu^2+7}{512((\nu^2-1)^3(\nu^2-4)}\left(\frac{1}{\hbar}\right)^8 + \frac{9\nu^4+58\nu^2+29}{4096(\nu^2-1)^5(\nu^2-4)(\nu^2-9)}\left(\frac{1}{\hbar}\right)^{12} + \ldots\right)$$

Mathieu, strong coupling

$N\hbar \gg 1$



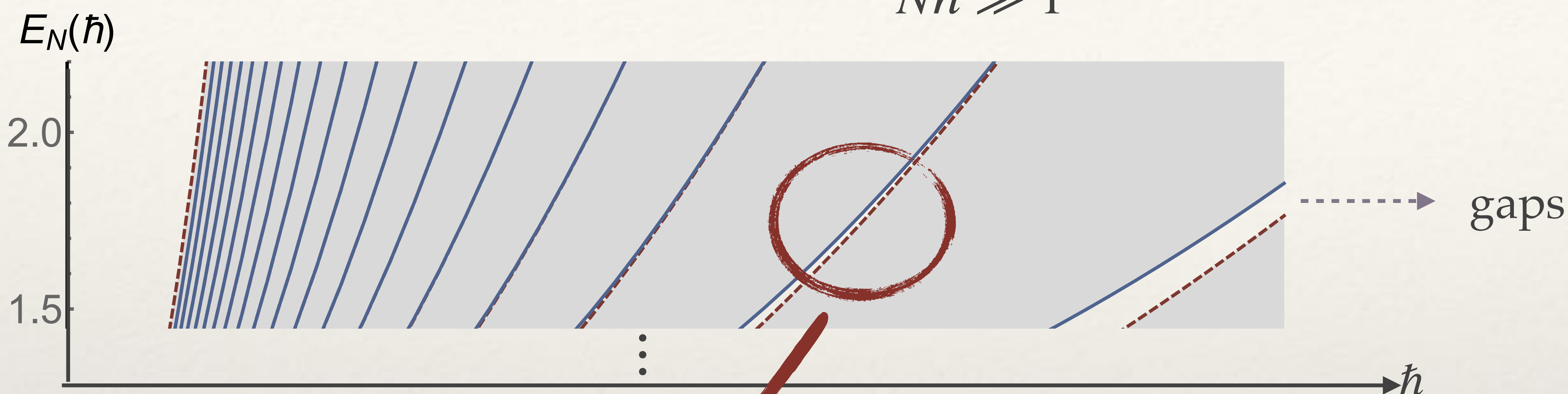


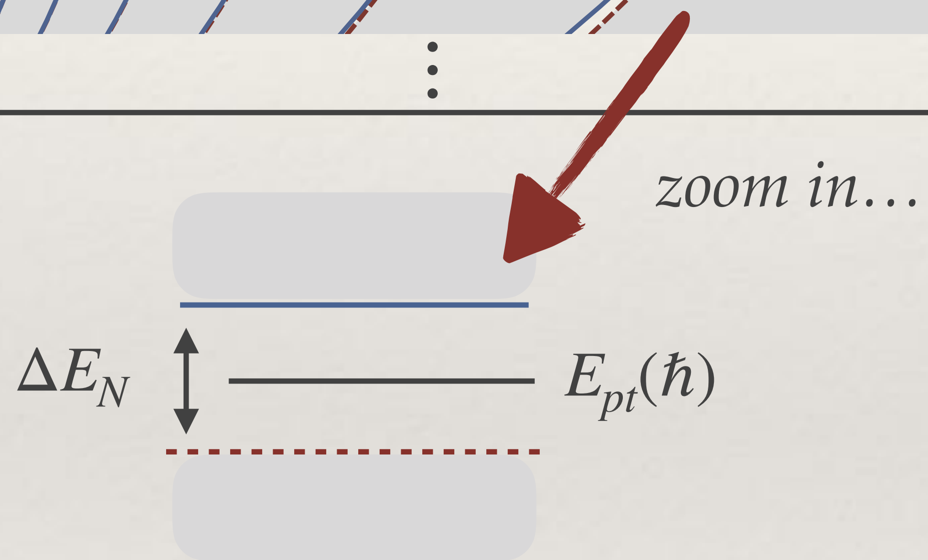
- Exponentially small
- Missed by Bohr-Sommerfeld

$$\Delta E_N = E_{N,even} - E_{N-1,odd} \approx \frac{1}{2^{N-2}\Gamma^2(N)\hbar^{2N-2}} \approx \frac{N\hbar^2}{\pi}\left(\frac{e}{\sqrt{2}N\hbar}\right)^{2N}$$

Mathieu, strong coupling

$N\hbar \gg 1$





- Exponentially small
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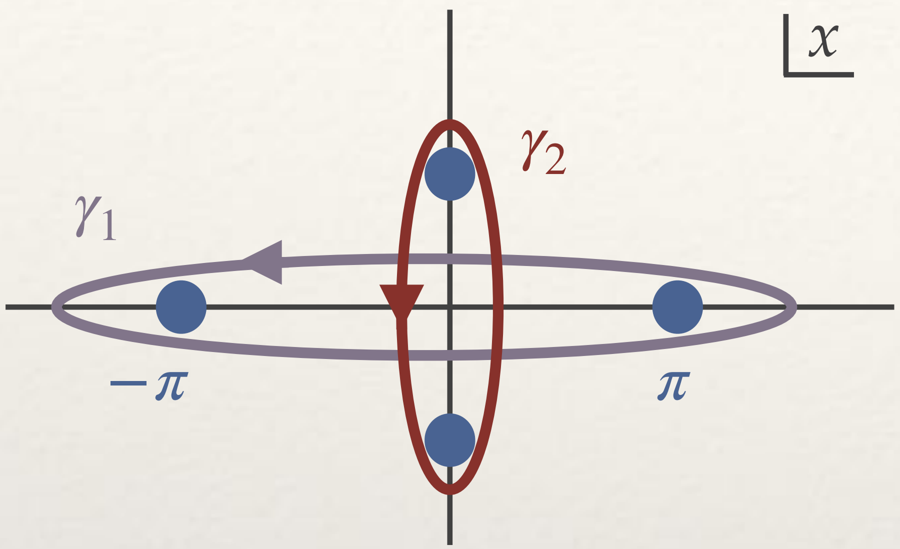
Just like… instantons…?

$$\Delta E_N = E_{N,even} - E_{N-1,odd} \approx \frac{1}{2^{N-2}\Gamma^2(N)\hbar^{2N-2}} \approx \frac{N\hbar^2}{\pi}\left(\frac{e}{\sqrt{2}N\hbar}\right)^{2N}$$

Mathieu, complex instantons



To leading order $E \approx \frac{\hbar^2 N^2}{2}$

$$\mathrm{Im} S_{\gamma_2}(E) = \sqrt{2}\pi(1-E)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-E\right) \approx -2\sqrt{2E}\left(\log(16E)-2\right)$$

$$\Delta E_N \approx \frac{1}{\pi}\frac{\partial E}{\partial N} e^{-\frac{1}{2\hbar}\mathrm{Im}\oint_{\gamma_2} P(x;\hbar)dx} \approx \frac{N\hbar^2}{\pi}\left(\frac{e}{2N^2\hbar^2}\right)^N$$

Generally:

"perturbative" *"1-instanton"*

$$E(\hbar) \sim \frac{\hbar^2}{2}\left(N^2 + \frac{1}{8(N^2-1)}\left(\frac{1}{\hbar}\right)^4 + \frac{5N^2+7}{512(N^2-1)^3(N^2-4)}\left(\frac{1}{\hbar}\right)^8 + \dots\right) \pm \frac{1}{2^{N-2}\Gamma^2(N)\hbar^{2N-2}}\left(1+\mathcal{O}(\hbar^{-4})\right) + \dots$$

$\dots$

bilinear identity

$$F_{inst}(\hbar, N) = \frac{\partial E_{pt}}{\partial N} e^{S_I \int_0^{\hbar} \frac{d\hbar}{\hbar^3}\left(\frac{\partial E_{pt}}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\hbar^2}{S_I}\right)}$$

[GB, Dunne, '15]

Complex instantons in QFT

Vacuum pair production with monochromatic electric field

$$E(t) = \mathscr{E}\cos(\Omega t)$$

Mathieu problem with $\hbar \Leftrightarrow \frac{\omega^2}{\mathscr{E}}$ *~frequency* $N \Leftrightarrow \frac{m_e}{\Omega}$ *~number of photons*

$$\hbar N \leftrightarrow \frac{m\Omega}{\mathscr{E}} := \gamma \quad \text{~"Keldysh adiabaticity parameter"}$$

Pair production rate:

$$e^{-\frac{m^2\pi}{\mathscr{E}} f\left(\frac{m\Omega}{\mathscr{E}}\right)} \sim \begin{cases} e^{-\frac{m^2\pi}{\mathscr{E}}}, & \gamma \ll 1 \\ \left(\frac{4m\Omega}{\mathscr{E}}\right)^{\frac{4m}{\Omega}}, & \gamma \gg 1 \end{cases}$$

- Static limit
- Schwinger pair production
- Tunnelling from Dirac sea
- ~band width

- Multi-photon limit
- Brézin-Itzykson
- Tunnelling from Dirac sea
- ~gap width

[GB, Dunne, '15]

Complex instantons, trans-asymptotics

$$E_{pt}(\hbar) \sim \frac{\hbar^2}{2}\left(\nu^2 + \frac{1}{8(\nu^2-1)}\left(\frac{1}{\hbar}\right)^4 + \frac{5\nu^2+7}{512(\nu^2-1)^3(\nu^2-4)}\left(\frac{1}{\hbar}\right)^8 + \frac{9\nu^4+58\nu^2+29}{4096(\nu^2-1)^5(\nu^2-4)(\nu^2-9)}\left(\frac{1}{\hbar}\right)^{12} + \ldots\right)$$

- Focus on a given pole, say ν=1,

$$E^{(\nu\sim1)}_{sing} \sim \frac{\hbar^2}{2}\left[\frac{1}{16(\nu-1)}\left(\frac{1}{\hbar}\right)^4 - \left(\frac{1}{1024(\nu^3-1)} - \frac{1}{1024(\nu-1)}\right)\left(\frac{1}{\hbar}\right)^8 + \ldots\right]$$

- Resum $\quad E^{(\nu\sim1)}_{sing} := \sum_{k=1}^{\infty}\left[f_k^{(\nu\sim1)}(z)\left(\frac{1}{\hbar}\right)^{4k-2}\right], \quad z := \frac{1}{\nu-1}\left(\frac{1}{\hbar}\right)^2$

- Expand around $\quad z \approx \infty \Rightarrow E_{1,\pm} = \frac{\hbar^2}{2}\left(1 \pm \frac{1}{\hbar} + \ldots\right)$

[Gorsky, Milekhin, Sopenko '17]

- Poles morph into brach points

can be interpreted as trans-asymptotics, see eg. book by [Costin]

Thank you!