Resurgence theory, analytical continuation of path integrals and ghost instantons

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October 29, 2013

with G.Dunne and M. Ünsal, arXiv:1308.1108, JHEP **10** (2013) 041

The resurgence of the ghost

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Motivation:

Can we make sense out of quantum field theory?

Can we define quantum field theory in continuum?

Dyson ('51) QED, 't Hooft ('79) QCD

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Can we make sense out of quantum field theory?

Can we define quantum field theory in continuum?

Dyson ('51) QED, 't Hooft ('79) QCD

Main issue: Perturbation theory is divergent.

Yes, even after regularization and renormalization!

A typical perturbative expansion in quantum field theory or quantum mechanics (ground state energy, matrix element, etc...):

$$f(g^2) = \sum_{n=0}^{\infty} c_n g^{2n}$$

diverges as $c_n \sim n!$ for large n $\Rightarrow$ asymptotic series

Instability at -g^2 $\Leftrightarrow$ zero radius of convergence (Dyson)

How can we sum an asymptotic series?

Borel summation

Introduce the Borel transform of the series $f(g^2)$:

$$\mathcal{B}[f](u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} u^n$$

This new series typically has finite radius of convergence.

Borel resummation of the original asymptotic series:

$$\mathcal{S}f(g^2) = \frac{1}{g^2} \int_0^{\infty} \mathcal{B}[f](u) e^{-u/g^2} du$$

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Borel resummation of the original asymptotic series:

$$\mathcal{S}f(g^2) = \frac{1}{g^2} \int_0^{\infty} \mathcal{B}[f](u) e^{-u/g^2} du$$

But $\mathcal{B}[f](u)$ in general can have singularities in the u plane.
They can be on $\mathbb{R}^+$

How to deal with those singularities?

Borel singularities

We can avoid the singularities on $\mathbb{R}^+$:

Lateral Borel sums:

$$\mathcal{S}_\theta f(g^2) = \frac{1}{g^2}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](u) e^{-u/g^2} du$$

In particular we can go above/below the singularity: $\theta = 0^\pm$
but this will leave us with an ambiguity: $\pm \mathcal{I}m[\mathcal{S}_0 f(g^2)]$
This is problematic if we are calculating real quantities.

In quantum mechanics/quantum field theory, this ambiguous imaginary part of pert. series is *exactly cancelled by instanton/anti-instanton ambiguity!*

$$\mathcal{I}m[\mathcal{S}_0 f(g^2)] + \mathcal{I}m[\mathcal{I}\bar{\mathcal{I}}] = 0$$

(Bogomolnyi,Zinn-Justin, Balitsky,Yung)

Borel singularities and Stokes' phenomenon

A function may have *different asymptotic expansions* depending on the direction of the expansion in complex plane.

An exponentially small correction might become comparable than the original series as one rotates in complex plane.

This new term "borns" when a Stokes line is crossed.

In Borel plane: Stokes line ⇔ Line of singularities

Resurgence

To keep track of all the Stokes jumps introduce *trans-series*

$$f(g^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k-1} a_{n,k,l}\, g^{2n}\, [\exp(-s/g^2)]^k\, [\log(-1/g^2)]^l$$

Resurgence: (Écalle,Dingle,Berry,Howls,Pham,...)
The analytical continuation in the Borel plane enforces relations between the lateral sums $\mathcal{B}_\theta[f]$. This translates into stringent constraints between the coefficients $a_{n,k,l}$.

perturbative ⇔ non-perturbative

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

Écalle

Resurgence and QFT

- Asymptotically free QFTs, QM, path integrals:
  Non-BPS objects ⇔ IR renormalons ⇔ confinement
  (Argyres,GB, Dunne,Cherman,Dorigoni,Ünsal)

- String theory, matrix models, Chern-Simons, localization...
  (Aniceto,Mariño,Pasquetti,Schiappa,Weiss,Vaz,Vonk)

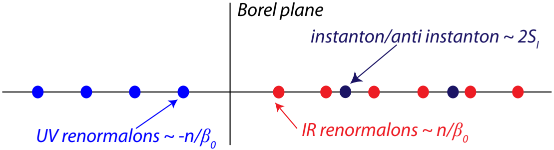
- Topological field theory (Garoufalidis, Costin)

- Path integral perspective (Kontsevich)

Aim: Continuum definition of QFT via exact encoding of the theory via trans-series.

Resurgence and QFT

In QFT: extra singularities due to momentum space integrals of Green's functions *"Renormalons"* ('t Hooft)



Claim: IR renormalons $\Leftrightarrow$ non-BPS defects with $S \sim \frac{1}{N}$

- semi-class. deformed Yang-Mills *(bions)* (Argyres,Ünsal)
- $\mathbb{C}P^N$ *(kink-instantons)* (Dunne,Ünsal)
- PCM *(fractons)* (Cherman,Dorigoni,Dunne,Ünsal)

Resurgence from the path integral perspective

Semiclassical expansion of a path integral

$$\mathcal{Z}(g^2) = \int \mathcal{D}\phi \, e^{-S[\phi]} \approx \sum_{\text{saddles}\, k} F_k(g^2)\, e^{-\frac{1}{g^2} S_k}$$

In QM with degenerate vacua or QFTs like Yang-Mills there are instantons as non-perurbative saddles.

Resurgence: The asymptotic expansions around different saddles of the path integral influence each other. ("functional Darboux theorem", extension of Berry-Howls to functional integrals)

Path integrals with complex saddles

In general, a path integral can have complex, even negative saddles: *“ghost instantons"*

$$\sum_k G_k(g^2)\, e^{+\frac{1}{g^2}|S_k|}$$

For sensible theories, the path integration exclude them. Should we just ignore them?

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$$\sum_k G_k(g^2)\, e^{+\frac{1}{g^2}|S_k|}$$

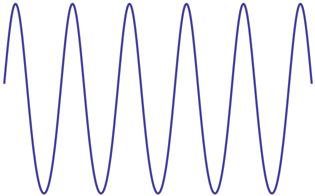
For sensible theories, the path integration exclude them.
Should we just ignore them? Certainly NO!

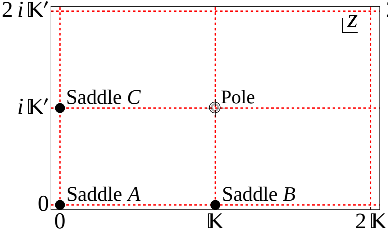
- They contribute to large order perturbation series.
- They can be important for extending the theory to complex/negative couplings and/or analytical continuation.
- They can play a role in quantum phase transitions.

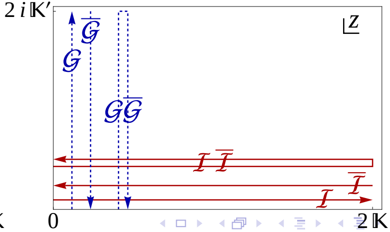
Path integrals with complex saddles

To illustrate the ideas, introduce a *doubly periodic* (periodic in real and imaginary axes) potential

$$V(z|m) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m)$$







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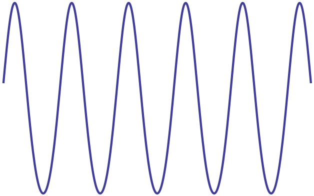

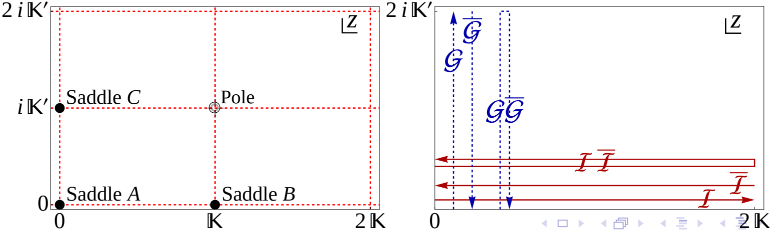

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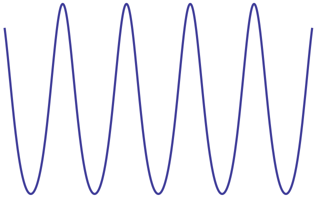

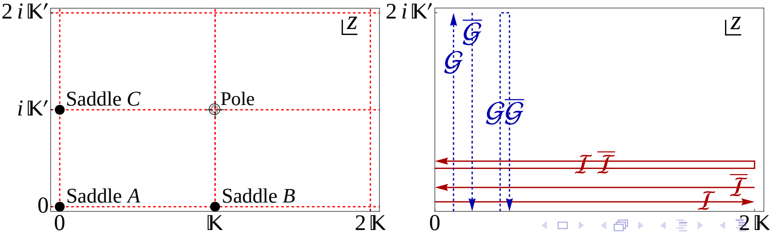

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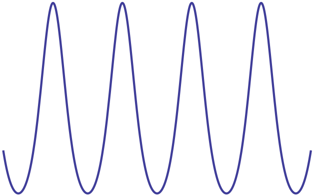

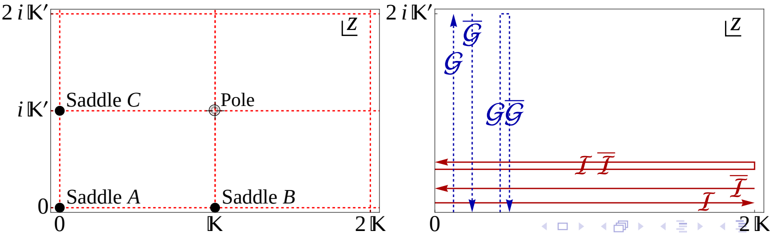

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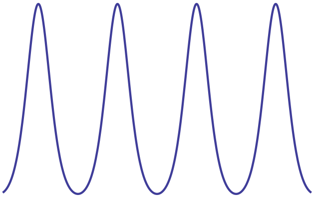

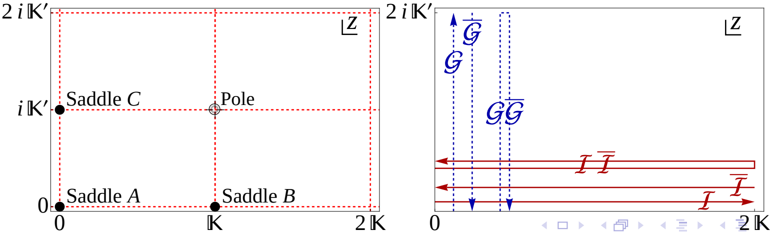

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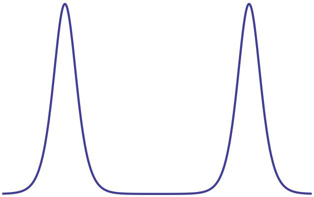

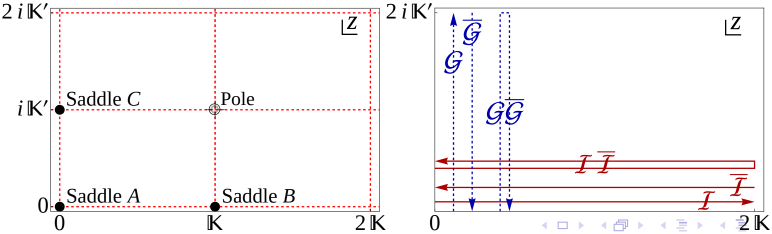

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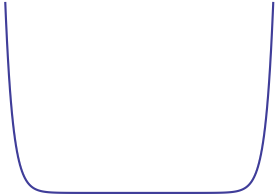

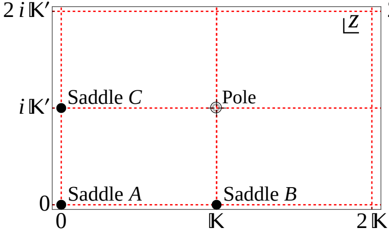

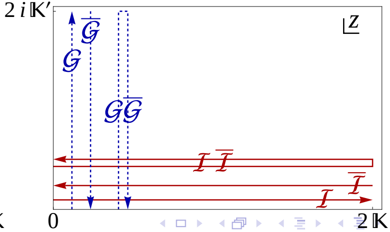

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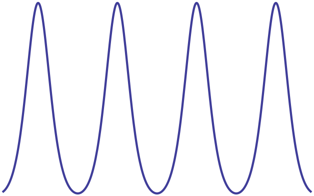

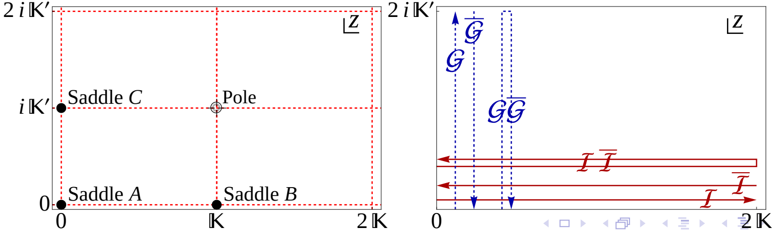

Path integrals with complex saddles

Self-duality and negative couplings:

$$V(z|m) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m)$$

The potential has a remarkable property:

$$V(z|m)|_{g^2} = V(z|m')|_{-g^2}, \qquad m' = 1 - m$$

For negative couplings, it can be mapped to *itself*.

Any perturbative series $\sum_n a_n(m) g^{2n}$ satisfies:

$$a_n(m) = (-1)^n a_n(1-m)$$

$m \to m' \Leftrightarrow g^2 \to -g^2 \Leftrightarrow$ alternating $\to$ non-alternating

Zero dimensional prototype

Partition function:

$$\mathcal{Z}(g^2|m) = \frac{1}{g\sqrt{\pi}} \int_{-\mathbb{K}}^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

Perturbative expansion

$$\mathcal{Z}(g^2|m)|_{pert} = \sum_{n=0}^{\infty} a_n(m) g^{2n}$$

is divergent for all m but is non-alternating for $m < 1/2$ and alternating for $m > 1/2$.

Puzzle: Alternating = Borel summable?
What about "instantons" ? They still exist for $m > 1/2$.

Zero dimensional prototype

Further puzzle:

saddle point on the path: $S_B = 1/m' \Rightarrow a_n \sim \frac{(n-1)!}{\pi S_B^{n+1/2}}$

Compare with the actual series:

Zero dimensional prototype

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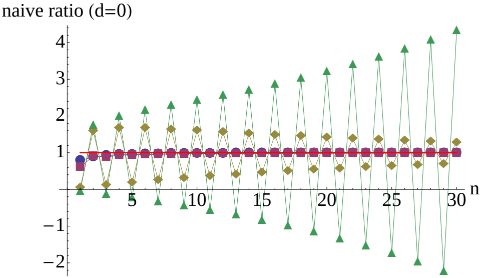

Disaster!

Zero dimensional prototype

Resolution:
There is another saddle outside the integration path!

$$S_C = -1/m \Rightarrow a_n \sim \frac{(n-1)!}{\pi}(S_B^{n+1/2} + S_C^{n+1/2})$$

Zero dimensional prototype

Resolution:

There is another saddle outside the integration path!

$$S_C = -1/m \Rightarrow a_n \sim \frac{(n-1)!}{\pi}(S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2})$$

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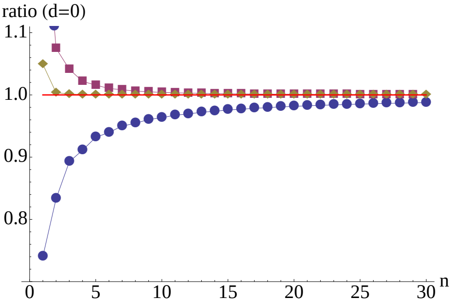
Improved asymptotics:



Perturbation series “sees" *all* the saddles!"

Zero dimensional prototype

The big picture:

- Associated with each critical point z_i, there is a unique integration cycle $\mathcal{J}_i$, called a *Lefschetz thimble*, along which the phase remains stationary.

- Around each saddle there is a contribution of the form:

$$\mathcal{I}^{(k)}(\xi|m) = \frac{1}{\sqrt{\pi}}\sqrt{\xi}\int_{\mathcal{J}_k} dz\, e^{-\xi\, \mathrm{sd}^2(z|m)}$$

- The expansions around saddle are connected via the

exact resurgence relation:

$$\mathcal{I}^{(A)}\left(\frac{1}{g^2}|m\right) = \frac{2}{2\pi i}\sum_{k\in\{B,C\}}\int_0^\infty \frac{dv}{v}\,\frac{1}{1-g^2\,v}\mathcal{I}^{(k)}(v|m)$$

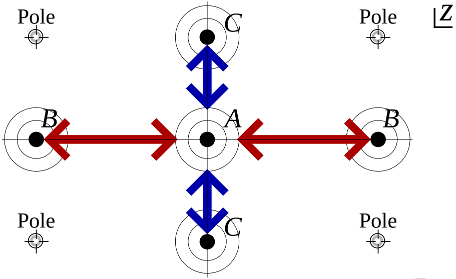
Zero dimensional prototype

The most general expansion is the three term trans-series

$$\mathcal{Z}_{\mathfrak{C}}(g^2|m) \equiv \sigma_A \Phi_A(g^2) + \sigma_B e^{-S_B/g^2} \Phi_B(g^2) + \sigma_C e^{-S_C/g^2} \Phi_C(g^2)$$

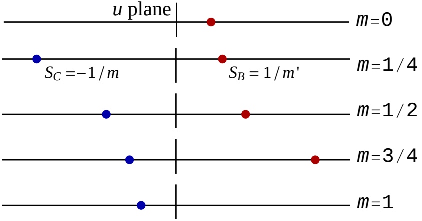
The coefficients of perturbative expansions are connected!

$$a_n^{(A)}(m) = \sum_{j=0} \frac{(n-j-1)!}{\pi} \left(\frac{a_j^{(B)}(m)}{S_B^{n-j}} + \frac{a_j^{(C)}(m)}{S_C^{n-j}} \right)$$



Zero dimensional prototype

The view from the Borel plane:



Distance in Borel plane, $\Delta S = S_i - S_j$ ("Relative action") controls the divergence of perturbation series Φ_j

In particular for $m > 1/2$:
closest singularity is on $\mathbb{R}^-$ $\Leftrightarrow$ alternating series Φ_A

But the sub-series associated with the sing. on $\mathbb{R}^+$ is non-alternating

Zero dimensional prototype

Cancellation of ambiguities and Stokes' phenomenon:

• For *real* path integration cycle, the partition function is a truncation of the trans-series.

• Borel-Écalle summation of this truncation renders a real, unambigious result:

$$\begin{aligned} \Phi_A(g^2) \pm i e^{-\frac{1}{m' g^2}} \Phi_B(g^2) \to & \quad \mathcal{S}_{0\pm} \Phi_A \pm i e^{-\frac{1}{m' g^2}} \mathcal{S}_{0\pm} \Phi_B \\ = & \quad \mathrm{Re}\mathcal{S}_0 \Phi_A + i \left(\mathrm{Im}\mathcal{S}_{0\pm} \Phi_A \pm e^{-\frac{1}{m' g^2}} \mathcal{S}_0 \Phi_B \right) \\ = & \quad \mathrm{Re}\mathcal{S}_0 \Phi_A \end{aligned}$$

Zero dimensional prototype

Going off the real axis; to the complex g^2 plane:

• Analytical continuation $\Leftrightarrow$ smooth deformation of the Lefshetz thimble

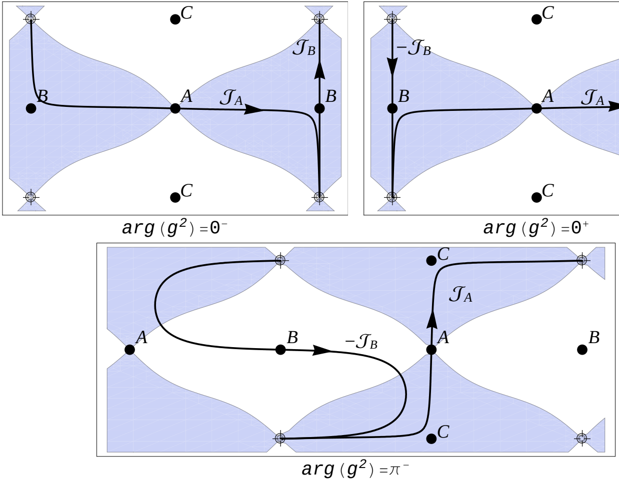
• Exponentially suppressed term at $\mathbb{R}^+$ becomes exponentially large at $\mathbb{R}^-$. $(e^{-\frac{1}{m' g^2}} = e^{\frac{1}{m'|g|^2}})$

Yet another puzzle:

The self-duality predicts a trans-series with terms $\mathcal{O}(1)$ and $\mathcal{O}(e^{-\frac{1}{m|g|^2}})$ for $g^2 < 0$ since $g^2 \to -g^2 \Leftrightarrow m \to m'$. But analytical continuation leads to an exponentially large term.

Zero dimensional prototype

Resolution: There is an independent Lefschetz thimble for $-g^2$ which leads to the self dual trans-series.



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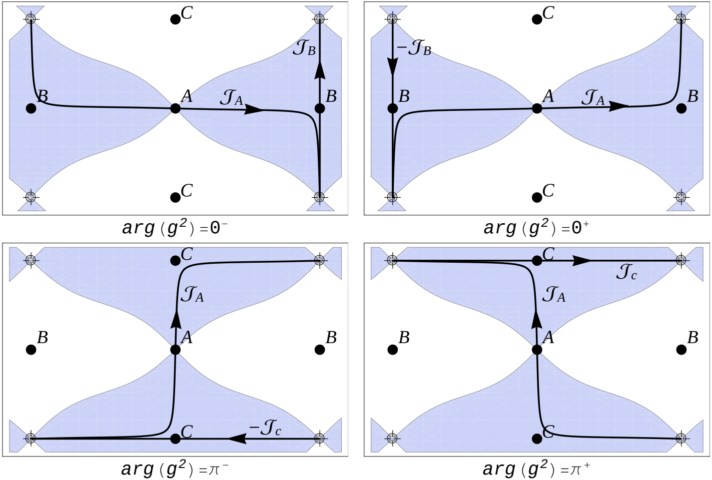
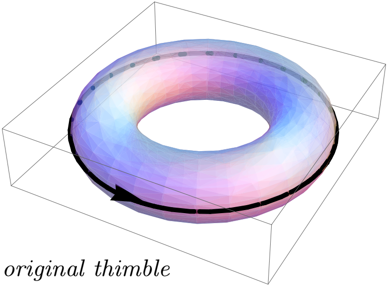
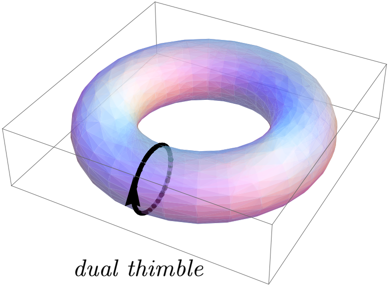


Zero dimensional prototype

Self-dual thimble and topology: The self dual thimble belongs to a different homotopy class than the original one. They cannot be related with analytical continuation.



original thimble



dual thimble

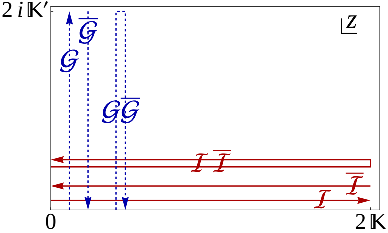
Physical interpretation: *Quantum phase transition*

Quantum mechanics

In QM we have the path integral:

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}\phi\, e^{-S[\phi]} = \int \mathcal{D}\phi\, e^{-\int d\tau \left(\frac{1}{4}\dot{\phi}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,\phi|m)\right)}$$

There are *real* and *ghost* instantons

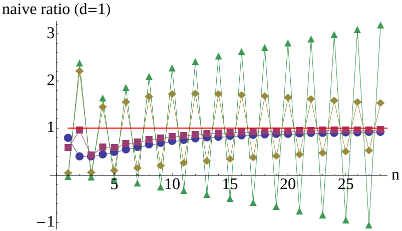


with actions

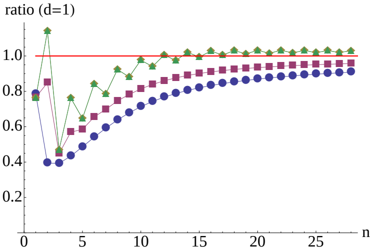
$$\frac{S_{\mathcal{I}}(m)}{g^2} = \frac{2\sin^{-1}(\sqrt{m})}{g^2\sqrt{mm'}} \geq \frac{2}{g^2} \quad , \quad \frac{S_{\mathcal{G}}(m)}{g^2} = \frac{2\sin^{-1}(\sqrt{m'})}{g^2\sqrt{mm'}} \leq -\frac{2}{g^2}$$

Quantum mechanics

The large order growth of perturbation theory



without ghost instantons



with ghost instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

Quantum mechanics

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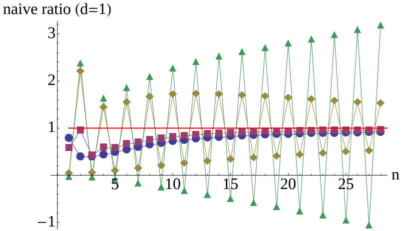

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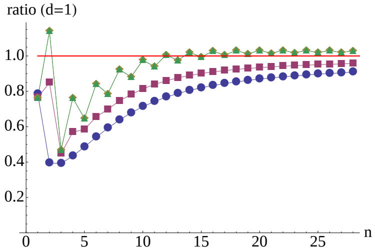

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$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

Notice the leading singularity is at $[\mathcal{I}\bar{\mathcal{I}}]$ or $[\mathcal{G}\bar{\mathcal{G}}]$

Quantum mechanics

The big picture:

- The vacuum "talks to" the topologically trivial sector:

$$\ldots \leftrightarrow [\mathcal{G}^2\bar{\mathcal{G}}^2] \quad \leftrightarrow \quad [\mathcal{G}\bar{\mathcal{G}}] \quad \leftrightarrow \quad \text{pert.vac} \quad \leftrightarrow \quad [\mathcal{I}\bar{\mathcal{I}}] \quad \leftrightarrow [\mathcal{I}^2\bar{\mathcal{I}}^2] \leftrightarrow \ldots$$

- The QM trans-series:

$$\mathcal{Z}(g^2|m) = \begin{cases} \Phi_0(g^2) + [\mathcal{I}\bar{\mathcal{I}}]_- \Phi_{[\mathcal{I}\bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2\bar{\mathcal{I}}^2]_- \Phi_{[\mathcal{I}^2\bar{\mathcal{I}}^2]}(g^2) + \ldots & -\pi < arg(g^2) < 0 \\ \Phi_0(g^2) + [\mathcal{I}\bar{\mathcal{I}}]_+ \Phi_{[\mathcal{I}\bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2\bar{\mathcal{I}}^2]_+ \Phi_{[\mathcal{I}^2\bar{\mathcal{I}}^2]}(g^2) + \ldots & 0 < arg(g^2) < \pi \end{cases}$$

- The ambiguities cancel ad-infinitum (resurgence!)

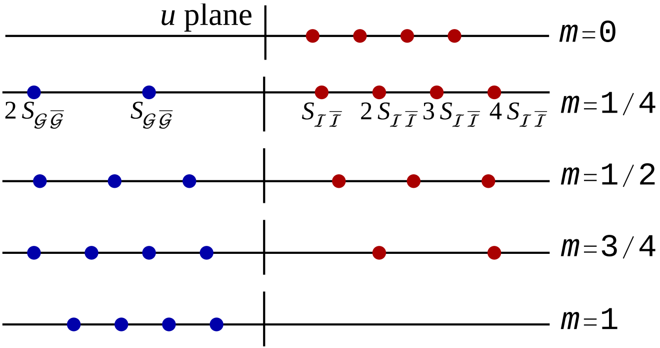
$$\text{Im}\left(\mathcal{S}_{0\pm}\Phi_0 + [\mathcal{I}\bar{\mathcal{I}}]_{0\pm}\text{Re}\mathcal{S}_0\Phi_{[\mathcal{I}\bar{\mathcal{I}}]}\right) = 0 \quad \text{up to} \quad \mathcal{O}(e^{-4S_I})$$

- Similar structure for one instanton, etc.. sector

$$\ldots \leftrightarrow [\mathcal{I}\mathcal{G}^2\bar{\mathcal{G}}^2] \quad \leftrightarrow \quad [\mathcal{I}\mathcal{G}\bar{\mathcal{G}}] \quad \leftrightarrow \quad [\mathcal{I}] \quad \leftrightarrow \quad [\mathcal{I}^2\bar{\mathcal{I}}] \quad \leftrightarrow [\mathcal{I}^3\bar{\mathcal{I}}^2] \leftrightarrow \ldots$$

Quantum mechanics

Borel plane:



Quantum mechanics

Negative coupling and the dual cycle:

- Analytical cont.: Instantons $\leftrightarrow$ ghosts $\leftrightarrow$ $e^{+c/|g|^2}$
- Dual thimble $\Leftrightarrow$ Pure imaginary paths $\Leftrightarrow$ No ghosts

$$g^2 > 0$$

$$\begin{array}{ccccccccccccc}
\ldots \leftrightarrow & -4S(m') & \leftrightarrow & -2S(m') & \leftrightarrow & 0 & \leftrightarrow & 2S(m) & \leftrightarrow & 4S(m) & \leftrightarrow \ldots \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\
\ldots \leftrightarrow & [\mathcal{G}^2\bar{\mathcal{G}}^2] & \leftrightarrow & [\mathcal{G}\bar{\mathcal{G}}] & \leftrightarrow & \text{pert.vac.} & \leftrightarrow & [\mathcal{I}\bar{\mathcal{I}}] & \leftrightarrow & [\mathcal{I}^2\bar{\mathcal{I}}^2] & \leftrightarrow \ldots \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\
\ldots \leftrightarrow & -4S(m) & \leftrightarrow & -2S(m) & \leftrightarrow & 0 & \leftrightarrow & 2S(m') & \leftrightarrow & 4S(m') & \leftrightarrow \ldots
\end{array}$$

$$g^2 < 0$$

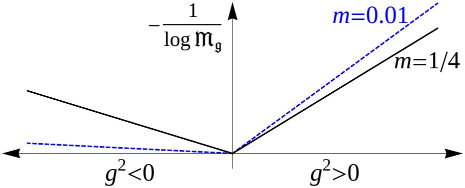
Quantum mechanics

Quantum phase transition:

self-duality in Hamiltonian picture:

$$H_m = -\frac{d^2}{dz^2} + \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m) \xrightarrow{g^2\to -g^2} H_{m'} = -\frac{d^2}{d\phi^2} + \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m')$$

mass gap/band-width $\mathfrak{M}_g \sim e^{-\frac{1}{m' g^2}} \xrightarrow{g^2\to -g^2} e^{-\frac{1}{m g^2}}$



non-analyticty at origin $\Leftrightarrow$ quantum phase transition

Conclusions

• Resurgence provides a way of bridging perturbative and non-perturbative world systematically (finer than topology). They are all parts of the grand trans-series that encodes all the information.

• Some saddles which are not included in the path integral may still contribute to *physical observables*. They might even be the leading objects that control the large order behavior.

• The relation between analytical continuation and resurgence is interesting. But there can be novel phenomena which are associated with cycles that can only be reached *non-analytically*. They can be associated with different quantum phases of the theory.

Future prospects

- Resurgence in Chern-Simons theories, Euler-Heienberg type expansions, dS/AdS effective actions, exact S-matrices..

- Constructing strong coupling expansions from resurgent trans-series, how to reconcile it with dualities such as S-duality. How about expansion in ‘t Hooft coupling? $\mathcal{N} = 4$ SYM?

- How to reconcile resurgence with OPE?

- Theories with complex Borel singularities (ABJM)

- Non-existence of renormalons in theories without topological molecules? (e.g. $\mathcal{N} = 2$)