Relativistic hydrodynamic fluctuations

Gökçe Başar

University of Illinois at Chicago

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with X. An*, M. Stephanov, H-U. Yee

* UIC grad student

Motivations

- hydrodynamics has reached a quantitative stage in heavy ion collisions
- thermal fluctuations are important
- HIC: " sweet spot" $N \sim \mathcal{O}(10^{2-4})$:
  - large enough for hydro
  - small enough that fluctuations are important for quantitative description
- approach to critical point
- other "small systems", condensed matter, unitary gases?

Hydrodynamics

hydrodynamics: universal macroscopic motion of fluids

fluid: continuous medium, composed of "infinitesimally small" cells of size a:

$$L \gg a \gg \ell_{\text{mic}}$$

Relativistic hydrodynamics

$$\partial_t \psi + \nabla(J[\psi]) = 0$$

hydrodynamic degrees of freedom: (ϵ, u^μ)

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu} + \Pi^{\mu\nu}$$

$$\Pi^{\mu\nu} = -2\eta \left(\theta^{\mu\nu} - \frac{1}{3}\Delta^{\mu\nu}\theta\right) - \zeta \Delta^{\mu\nu}\theta$$

↳ shear viscosity ↳ bulk viscosity

$$\theta^{\mu\nu} = \tfrac{1}{2}\left(\partial_\perp^\mu u^\nu + \partial_\perp^\nu u^\mu\right), \quad \theta = \theta^\mu_\mu \quad \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \quad \partial_\perp{}^\mu = \Delta^{\mu\nu}\partial_p \nu$$

gradients are assumed to be small:

$$\frac{\eta}{\epsilon + p}\partial \sim \frac{\eta}{\epsilon + p}\frac{1}{L} \equiv \frac{\eta}{\epsilon + p}k \equiv \gamma_\eta k \sim \frac{k}{T} \ll 1$$

Relativistic hydrodynamics

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Fluctuations

hydrodynamic variables (ϵ, u^μ) fluctuate

$$(\check{\epsilon}, \check{u}^\mu) = (\epsilon, u^\mu) + (\delta\epsilon, \delta u^\mu)$$

Fluctuation-dissipation theorem:

dissipation: $\Pi^{\mu\nu}$ $\Rightarrow$ fluctuation: $S^{\mu\nu}$ (noise)

Stochastic approach:

$$\begin{aligned}
\check{T}^{\mu\nu} &= T^{\mu\nu}_{ideal}(\check{\epsilon}, \check{u}) + \Pi^{\mu\nu}(\check{\epsilon}, \check{u}) + S^{\mu\nu} \\
\langle S^{\mu\nu}(x) \rangle &= 0 \\
\langle S^{\mu\nu}(x) S^{\lambda\kappa}(x') \rangle &= 2T \left(\eta(\Delta^{\mu\kappa}\Delta^{\nu\lambda} + \Delta^{\mu\lambda}\Delta^{\nu\kappa}) \right. \\
&\quad \left. + (\zeta - 2/3\eta)\Delta^{\mu\nu}\Delta^{\lambda\kappa} \right) \delta^{(4)}(x - x')
\end{aligned}$$

$\partial_\mu \check{T}^{\mu\nu} = 0$, relativistic Navier-Stokes-Langevin eqn.

(stochastic PDE)

Fluctuations: infinite noise

problem with stochastic equation:

$$\langle S(x)S(x')\rangle \propto \delta^{(3)}(x-x')\delta(t-t') \propto 1/a^3\Delta t$$

$$\text{noise} \sim 1/(\sqrt{\Delta t}\, a^{3/2}) \sim \Lambda^{3/2}/\sqrt{\Delta t}$$

($\Lambda \sim a^{-1}$: "UV cutoff scale")

large magnitude, numerically difficult to implement

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Deterministic approach

alternatively:

(Akamatsu, Mazeliauskas, Teaney '16)

(also Martinez, Schaefer '18, Andreev '70s (non relativistic))

- fluctuations $\rightarrow W = \langle \phi\phi \rangle$
- instead of stochastic relativistic NSL equation work with deterministic relativistic Navier Stokes + fluctuation evolution equation for W

$$\begin{aligned} \partial_t \psi &= -\nabla(\text{flux}[\psi, W]), \\ \partial_t W &= -\mathbb{M}^T W - W\mathbb{M}^T - 2Tw\mathbb{Q} \end{aligned}$$

- analogous to Langevin vs. Fokker Planck

Outline of this talk

- derive the deterministic evolution equation for fluctuations for *arbitrary background*, maintaining Lorentz covariance
- isolate the divergences due to infinite noise, carry out renormalizations analytically

Fluctuations

gradients drive the system slightly out of equilibrium,

scale of gradients: $L, \tau \sim L/c_s$

equilibration: via diffusion, takes time $l_{eq} \sim \sqrt{\gamma_\eta \tau} \ll L$

Fluctuations

$y \equiv$ size of fluctuations

$$\ell_{\text{mic}} \ll a \ll y \ll L$$

$y < l_{eq} \Rightarrow$ equilibrated $\qquad y > l_{eq} \Rightarrow$ not-equilibrated

Fluctuations

$\frac{1}{y} \sim q \equiv$ wave number of fluctuations

$$T \gg \Lambda \gg q \gg k$$

equilibration scale: $q_{eq} \sim \frac{1}{l_{eq}} \sim \sqrt{c_s k / \gamma_\eta}$

$q > q_{eq} \Rightarrow$ equilibrated $\quad$ $q < q_{eq} \Rightarrow$ not-equilibrated

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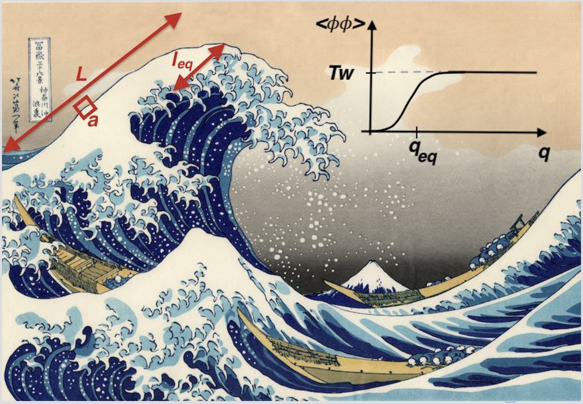


Fluctuations

$q > q_{eq} \Rightarrow$ equilibrated $\qquad q < q_{eq} \Rightarrow$ not-equilibrated

Fluctuation evolution equation: relaxation of non-equilibrium modes

Corrections to background flow

fluctuations $\Rightarrow$ corrections to background (average) flow

$$(\breve{\epsilon}, \breve{u}^\mu) = (\epsilon, u^\mu) + (\delta\epsilon, \delta u^\mu), \qquad \langle\delta\epsilon\rangle = \langle\delta u^\mu\rangle = 0$$

linear fluctuations $\Rightarrow$ non-linear terms in $T^{\mu\nu}$

$$\begin{aligned} \breve{u}^\mu \breve{u}^\nu &= u^\mu u^\nu + (u^\mu \delta u^\nu + u^\mu \delta u^\nu) + \delta u^\mu \delta u^\nu \\ p(\breve{\epsilon}) &= p(\epsilon) + c_s^2 \delta\epsilon + \frac{1}{2}\frac{dc_s}{d\epsilon}\delta\epsilon^2 + \ldots \end{aligned}$$

after some redefinitions, in Landau frame:

$$\langle \breve{T}^{\mu\nu} \rangle = T^{\mu\nu}(\phi) + a\,\langle\delta\epsilon\delta\epsilon\rangle\Delta^{\mu\nu} + b\,\langle\delta u^\mu \delta u^\nu\rangle$$

$\partial_\mu \langle \breve{T}^{\mu\nu}\rangle = 0$ & evolution eq for $\langle\phi\phi\rangle$: closed system of eqns.

Fluctuations: Wigner function

distribution of fluctuations: Wigner function

$$W_{AB}(t;\mathbf{x},\mathbf{q}) = \int d^3y\, e^{-i\mathbf{q}\cdot\mathbf{y}}\, \langle \delta\phi_A(t;\mathbf{x}+\mathbf{y}/2)\delta\phi_B(t;\mathbf{x}-\mathbf{y}/2)\rangle$$

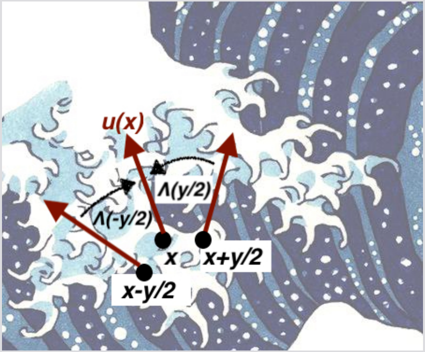
expect some from of kinetic equation with relaxation term: $\mathcal{C}[W]$

$$\frac{\partial W}{\partial t} + \dot{\mathbf{x}}\cdot\frac{\partial W}{\partial \mathbf{x}} + \dot{\mathbf{q}}\cdot\frac{\partial W}{\partial \mathbf{q}} = \mathcal{C}[W]$$

not quite, Lorentz covariance complicates things

- equal time? only have local u^μ
- $\mathbf{q}$: Fourier trans. of the local space-like $\mathbf{y}$-space, has to be defined locally via projection $\Delta^{\mu\nu}(x)$

Covariant Wigner function



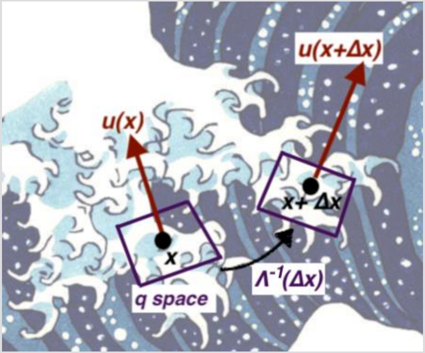
$$\bar{G}_{AB}(x,y) = \Lambda_A{}^C(y/2)\,\langle \phi_C(x+y/2)\phi_D(x-y/2)\rangle\,\Lambda_B{}^D(-y/2)$$

remains transverse: $u^A(x)\bar{G}_{AB}(x,y) = u^B(x)\bar{G}_{AB}(x,y) = 0$

covariant Wigner function:

$$W_{AB}(x,q) = \int d^4y\,\delta(u(x)\cdot y)\,e^{-iq\cdot y}\,\bar{G}_{AB}(x,y)$$

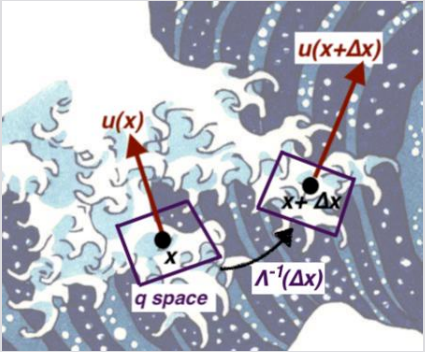
Confluent derivative



$$\Delta x \cdot \bar{\nabla} W_{AB}(x,q) = \Lambda_A{}^C \Lambda_B{}^D \bar{G}_{CD}(x+\Delta x, \Lambda^{-1} q) - W_{AB}(x,q)$$

removes change in W due to the difference in $u(x)$, measures the change in "internal" state of fluctuations

Confluent derivative



$$\bar{\nabla}_\mu W_{AB} = \partial_\mu W_{AB} - \bar{\omega}_{\mu A}^{\ C} W_{CB} - \bar{\omega}_{\mu B}^{\ C} W_{AC} + \mathring{\omega}_{\mu a}^{\ b}\, q_b \frac{\partial}{\partial q_a} W_{AB}$$

similar to covariant derivative

boost generator ⇔ "spin connection"

$$(\bar{\omega}_{\lambda\mu}^{\nu} = u_\mu \partial_\lambda u^\nu - u^\nu \partial_\lambda u_\mu, \quad \mathring{\omega}_{\lambda a}^{\ b} = e_\mu^b \partial_\lambda e_a^\mu)$$

Fluctuation evolution equation

$W_{AB}(x,q)$ satisfies an evolution equation that can be derived from stochastic hydrodynamics

sketch of the derivation

$$\partial_\mu \breve{T}^{\mu\nu} = 0$$

$$\Downarrow$$

$$u \cdot \partial \phi_A = -\mathbb{M}_{AB}\phi_B - \xi_A$$

$$\Downarrow$$

$$u \cdot \bar{\nabla} G_{AB}(x,y) = -\tilde{\mathbb{M}}_{AC} G_{CB}(x,y) - \tilde{\mathbb{M}}_{BC} G_{AC}(x,y) + 2\,Tw\mathbb{Q}^{(y)}_{AB}\delta^3(y_\perp)$$

$$(\mathbb{Q}^{(y)}_{AB} \sim \gamma \partial_\perp^{(y)2})$$

Fluctuation evolution equation

four modes of W_{AB}: 1 sound, 3 transverse

sound mode:

$$u \cdot \bar{\nabla} W + c_s \hat{q} \cdot \bar{\nabla} W + f^\mu \frac{\partial W_\pm}{\partial q_\mu} + B W = -\gamma_L q^2 (W - Tw)$$

$$\text{inertial forces} : f^\mu = -\left(c_s - \frac{\dot{c}_s}{c_s} \right) |q| a_\mu - (\partial_{\perp\mu} u_\nu) q^\nu - 2 c_s^2 q^\lambda \omega_{\lambda\mu}$$

$$\text{background} : B = \left((1 + c_s^2 + \dot{c}_s)\theta + \theta_{\mu\nu} \hat{q}^\mu \hat{q}^\nu \pm \frac{1 + 2c_s^2}{c_s} \hat{q} \cdot a \right)$$

Fluctuation evolution equation

sound mode:

$$\mathcal{N} = \frac{W}{w\, c_s |q|}$$: distribution function of phonons

$$(u+v)\cdot\bar{\nabla}\mathcal{N} + f^{\mu}\frac{\partial \mathcal{N}}{\partial q_{\perp\mu}} = -\gamma_L q^2(\mathcal{N} - T/E)$$

inertial forces:

$$f^{\mu} = -\left[E(a_{\mu} + 2v^{\nu}\omega_{\nu\mu}) + q_{\perp\nu}\partial_{\perp\mu}u^{\nu} + \bar{\nabla}_{\perp\mu}E\right]$$

$$E = c_s|\hat{q}|, \quad v^{\mu} = c_s\frac{\partial E}{\partial \hat{q}^{\mu}}$$

Fluctuation evolution equation

sound mode:

$$(u+v)\cdot\bar{\nabla}\mathcal{N} + f^{\mu}\frac{\partial\mathcal{N}}{\partial q_{\perp\mu}} = -\gamma_L q^2(\mathcal{N} - T/E)$$

Liouville operator $\mathcal{L}[\mathcal{N}] \equiv \dot{x}^{\mu}\frac{\partial\mathcal{N}}{\partial x^{\mu}} + \dot{q}^{\mu}\frac{\partial\mathcal{N}}{\partial q_{\mu}}$

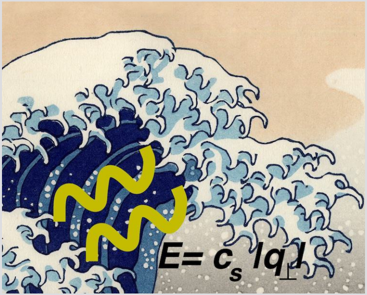
$$\dot{x} = u+v, \quad \dot{q}^{\mu} = -\partial^{\mu}E - p_{\nu}\partial^{\mu}u^{\nu}, \quad q^{\mu} = q_{\perp}^{\mu} + Eu^{\mu}$$

at constant $q_{\perp}$: $\partial_{\mu} = \bar{\nabla}_{\mu} + \partial_{\mu}q_{\perp}^{\nu}\frac{\partial}{\partial q_{\perp}^{\nu}} = \bar{\nabla}_{\mu} - e^{a\nu}\partial_{\mu}e_{\nu}^{b}q_a\frac{\partial}{\partial q^b}$

$$\Rightarrow \mathcal{L}[\mathcal{N}] = (u+v)\cdot\bar{\nabla}\mathcal{N} + f^{\mu}\frac{\partial\mathcal{N}}{\partial q_{\perp\mu}}$$

Fluctuation evolution equation

sound mode:



evolution equation ⇔ phonon kinetic equation with background!

$$\mathcal{L}[\mathcal{N}] = -\gamma_L q^2(\mathcal{N} - T/E)$$

equilibrium distribution: T/E: high T limit of Bose-Einstein

Fluctuation evolution equation

four modes of W_{AB} : 1 sound, 3 transverse

transverse modes:

$\widehat{W}_{ij}$: 2×2 matrix, $\quad i, j = 1, 2, \quad t^{(i)} \cdot u = 0, t^{(i)} \cdot q = 0$

$$\mathcal{L}[\widehat{W}] = -2q^2\gamma_\eta(\widehat{W} - Tw\widehat{\mathbb{1}}) - \left\{\widehat{K}, \widehat{W}\right\} + \left[\widehat{\Omega}, \widehat{W}\right]$$

$$\widehat{K}^{ij} \equiv \tfrac{1}{2}\theta\,\delta^{ij} + \theta^{\mu\nu} t_\mu^{(i)} t_\nu^{(j)}, \quad \text{and} \quad \widehat{\Omega}^{ij} \equiv \omega^{\mu\nu} t_\mu^{(i)} t_\nu^{(j)}$$

(purely diffusive modes with $E = 0$)

Renormalization

corrections to $\langle T^{\mu\nu} \rangle \propto \langle \delta\phi_A(x)\delta\phi_B(x)\rangle = \int \frac{d^3q}{(2\pi)^3} W_{AB}(x, \mathbf{q})$

"infinite noise": modes with $q \sim \Lambda \gg q_{eq}$

large q expansion

$$\text{large } q\text{:} \quad W_{AB}(x,\mathbf{q}) = Tw\Delta_{AB} + \frac{Tw}{\gamma \mathbf{q}^2} \times \partial u + \mathcal{O}(\mathbf{q}^{-4})$$

$$\Rightarrow \int \frac{d^3q}{(2\pi)^3} W_{AB}(x,\mathbf{q}) = \frac{Tw\Lambda^3}{6\pi^2}\Delta_{AB} + \frac{Tw}{6\pi^2}\frac{\Lambda}{\gamma} \times \partial u + \text{finite}$$

$Tw\Lambda^3 \Rightarrow$ equilibrium fluctuations

$Tw\frac{\Lambda}{\gamma} \times \partial u \Rightarrow$ non-equilibrium fluctuations

Renormalization: non equilibrium part

$$\Rightarrow \int \frac{d^3q}{(2\pi)^3} W_{AB}(x,\mathbf{q}) \;=\; \frac{Tw\Lambda^3}{6\pi^2} + \frac{Tw}{6\pi^2}\frac{\Lambda}{\gamma} \times \partial u + \text{finite}$$

renormalization of shear and bulk viscosity:

$$\eta_R \;=\; \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_\zeta + 4/3\gamma_\eta} + \frac{7}{2\gamma_\eta}\right)$$

$$\zeta_R \;=\; \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_\zeta + 4/3\gamma_\eta}(1-3c_s^2+\dot{c}_s)^2 + \frac{2}{\gamma_\eta}(1-3c_s^2)^2\right)$$

$$\dot{c}_s := d\log c_s / d\log s$$

sound modes contributing to transport:
renormalization in a Wilsonian sense, “hydro loops"

(Kovtun, Moore, Romatschke, ...)

Finite terms

$$\widetilde{G}_{AB}(x) = \int d^3q \left(W_{AB}(x, \mathbf{q}) - Tw - \frac{Tw}{\gamma \mathbf{q}^2} \times \partial u \right) = \text{“finite"}$$

$$\Downarrow$$

$$\begin{aligned} \langle \check{T}^{\mu\nu}_R(x) \rangle &= \epsilon u^\mu u^\nu + p(\epsilon)\Delta^{\mu\nu} + \Pi^{\mu\nu} \\ &+ \frac{1}{w}\left(\dot{c}_s \widetilde{G}_{ee}(x) - c_s^2 \widetilde{G}^\lambda_\lambda(x) \right) \Delta^{\mu\nu} + \frac{1}{w} \widetilde{G}^{\mu\nu}(x) \end{aligned}$$

with renormalized η, ζ, e.o.s.

full evolution of the system

$$\partial_\mu \langle \check{T}^{\mu\nu}_R(x) \rangle = 0, \quad u \cdot \bar{\nabla} W = -\mathbb{M} W - W \mathbb{M}^T - 2Tw\mathbb{Q}$$

Long-time tails

typical fluctuation mode: $q \sim q_{eq} \sim \sqrt{\gamma k}$

contribution to gradient expansion:

$$\widetilde{G}_{AB} \sim \int d^3q \widetilde{W}(x, \mathbf{q}) \sim (\gamma k)^{3/2}, (\gamma\omega)^{3/2}$$

alternatively: $\frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{V/\xi}} \sim \frac{1}{\sqrt{L^3/T^3}} \sim \left(\frac{k}{T}\right)^{3/2}$

- non-analytic terms, do not exist in ordinary gradient expansion
- more important than second order hydro, $\mathcal{O}(k^2)$

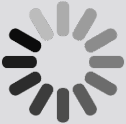
Conclusions

- deterministic formalism: fluctuations $\Leftrightarrow$ additional modes W_{AB}
- fully covariant covariant, equations for a general background

$$\partial_\mu \langle \check{T}^{\mu\nu} \rangle = 0 \quad + \quad \text{evolution eqn for } W$$

- renormalizations can be done analytically
- finite part can be implemented numerically

Future directions



- conserved charges *(in progress)*
- numerical implementation
- connection to Hydro+
- effects of shear, vorticity
- understanding long-time tails
- relations to effective action, Schwinger Keldysh based approaches (Kovtun, Loganayagam et al., Liu, Glorioso, ...)

$$
\begin{aligned}
u \cdot \bar{\nabla} W(x;q) &= -\left[i\mathbb{L}^{(q)} + \mathbb{K}^{(a)}, W\right] - \left\{\frac{1}{2}\bar{\mathbb{L}}^{(x)} + \mathbb{Q}^{(q)} + \mathbb{K}^{(s)}, W\right\} + (\partial \cdot u) W \\
&\quad + 2Tw\mathbb{Q}^{(q)} + (\partial_{\perp\lambda} u_\mu) q^\mu \frac{\partial W}{\partial q_\lambda} + \frac{1}{2} a_\lambda \left\{ \left(1 - \frac{\dot{c}_s}{c_s^2}\right) \mathbb{L}^{(q)}, \frac{\partial W}{\partial q_\lambda} \right\} \\
&\quad + \frac{\partial}{\partial q_\lambda} \left(\{\mathbb{\Omega}_\lambda^{(s)}, W\} + [\mathbb{\Omega}_\lambda^{(a)}, W] - \frac{1}{4}[\mathbb{H}_\lambda, [\mathbb{L}^{(q)}, W]] \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{L}^{(q)} &\equiv c_s \begin{pmatrix} 0 & q_\nu \\ q_\mu & 0 \end{pmatrix}, \quad \bar{\mathbb{L}}^{(x)} \equiv c_s \begin{pmatrix} 0 & \bar{\nabla}_{\perp\nu} \\ \bar{\nabla}_{\perp\mu} & 0 \end{pmatrix}, \\
\mathbb{Q}^{(q)} &\equiv \begin{pmatrix} 0 & 0 \\ 0 & \gamma_\eta \Delta_{\mu\nu} q^2 + \left(\gamma_\zeta + \frac{1}{3}\gamma_\eta\right) q_\mu q_\nu \end{pmatrix} \\
\mathbb{K}^{(s)} &\equiv \begin{pmatrix} (1 + c_s^2 + \dot{c}_s)\theta & \frac{1}{2c_s}(1 + 2c_s^2)\, a_\nu \\ \frac{1}{2c_s}(1 + 2c_s^2)\, a_\mu & \Delta_{\mu\nu}\theta + \theta_{\mu\nu} \end{pmatrix}, \quad \mathbb{K}^{(a)} \equiv \begin{pmatrix} 0 & -\frac{1 - c_s^2 - \dot{c}_s}{2c_s} a_\nu \\ \frac{1 - c_s^2 - \dot{c}_s}{2c_s} a_\mu & -\omega_{\mu\nu} \end{pmatrix} \\
\mathbb{H}_\lambda &\equiv c_s \begin{pmatrix} 0 & \partial_\nu u_\lambda \\ \partial_\mu u_\lambda & 0 \end{pmatrix}, \quad \mathbb{\Omega}_\lambda^{(s)} \equiv \frac{c_s^2}{2} \begin{pmatrix} 2\omega_{\kappa\lambda} q^\kappa & 0 \\ 0 & \omega_{\mu\lambda} q_\nu + \omega_{\nu\lambda} q_\mu \end{pmatrix}, \\
\mathbb{\Omega}_\lambda^{(a)} &\equiv \frac{c_s^2}{2} \begin{pmatrix} 0 & 0 \\ 0 & \omega_{\mu\lambda} q_\nu - \omega_{\nu\lambda} q_\mu \end{pmatrix}
\end{aligned}
$$