Reconstructing the critical equation of state from Taylor series

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FunQCD: from first principles to effective theories
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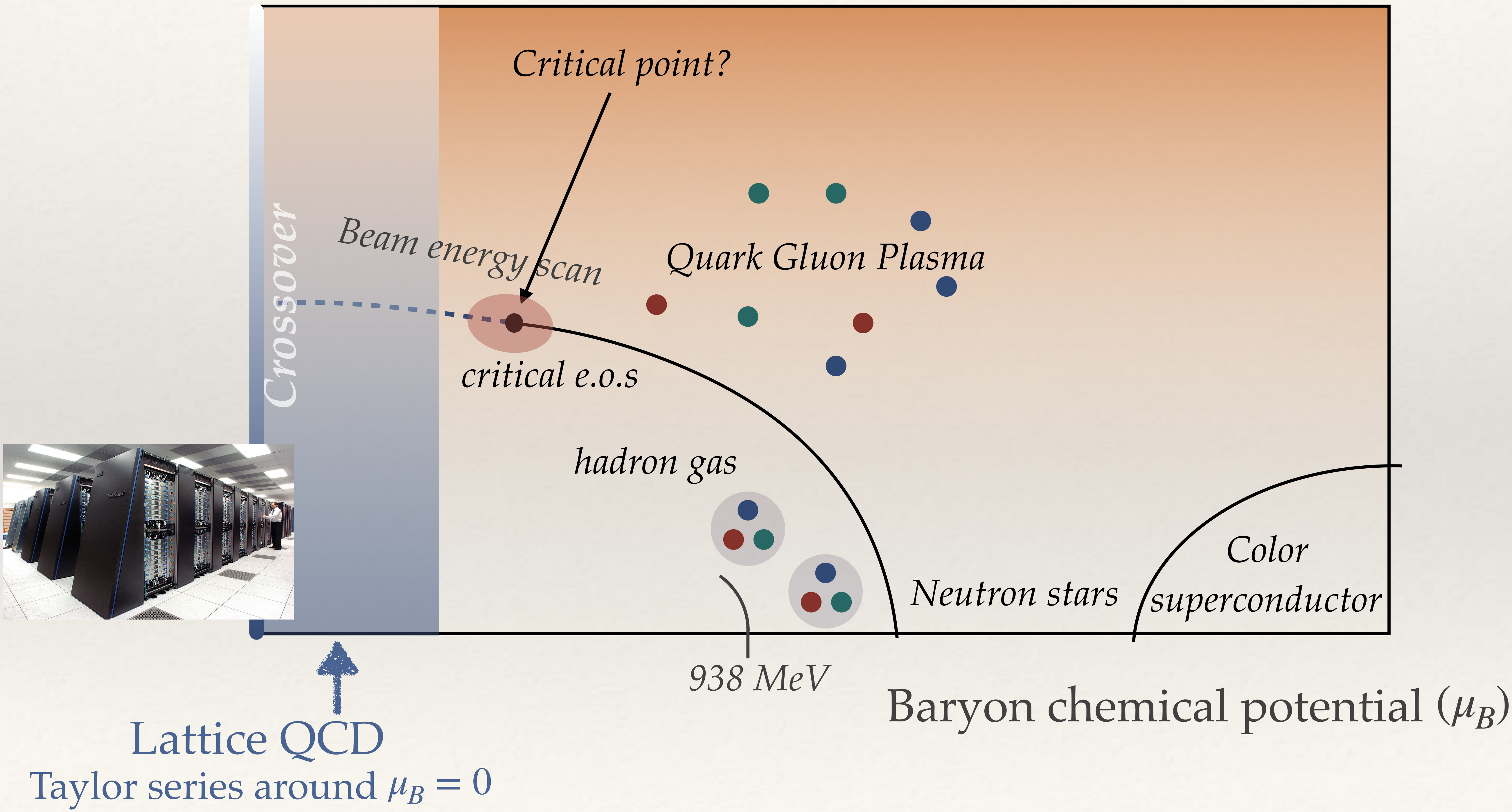
Motivations

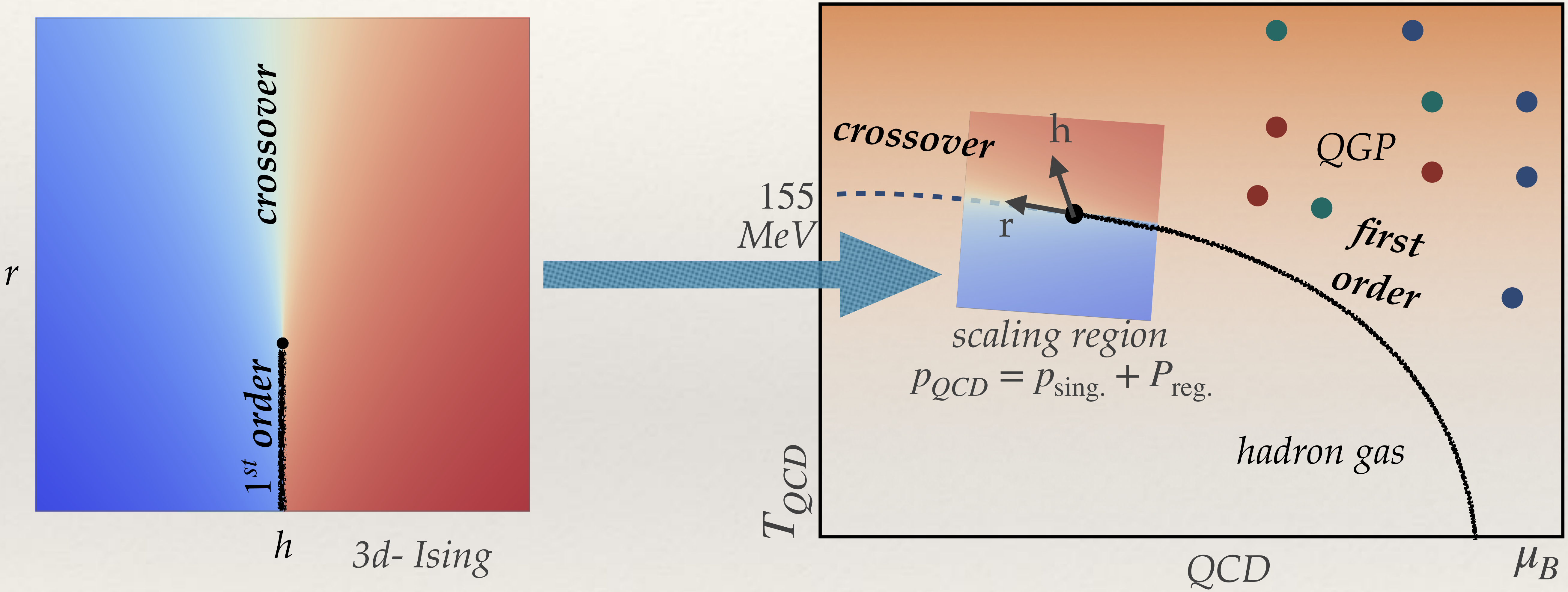


Motivations

BEST collaboration

$$\begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} := M \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$

see the talks by
Mroczek, Parotto , Stafford



Given the e.o.s. as truncated Taylor series around μ=0, what can we say about *the critical e.o.s* ?

Gross-Neveu Model

$$S = \int d^2x \sum_{a=1}^{N_F} \left(i\bar{\psi}_a(\partial\!\!\!/ - m_q)\bar{\psi}_a + \frac{g^2}{2}(\bar{\psi}_a\psi_a)^2 \right)$$

[Gross, Neveu, '74]

- Solvable in large N_f limit (mean field is exact at $N_f{=}\infty$)
- Asymptotically free, dimensional transmutation
- Chiral symmetry breaking
- Toy model for QCD

$$\frac{\pi}{Ng^2} = \log\frac{\Lambda}{m}$$

Dimensional transmutation

$$\gamma \equiv \frac{\pi}{Ng^2}\frac{m_q}{m} = \log\frac{m[m_q]}{m[0]}$$

Explicit χSB parameter

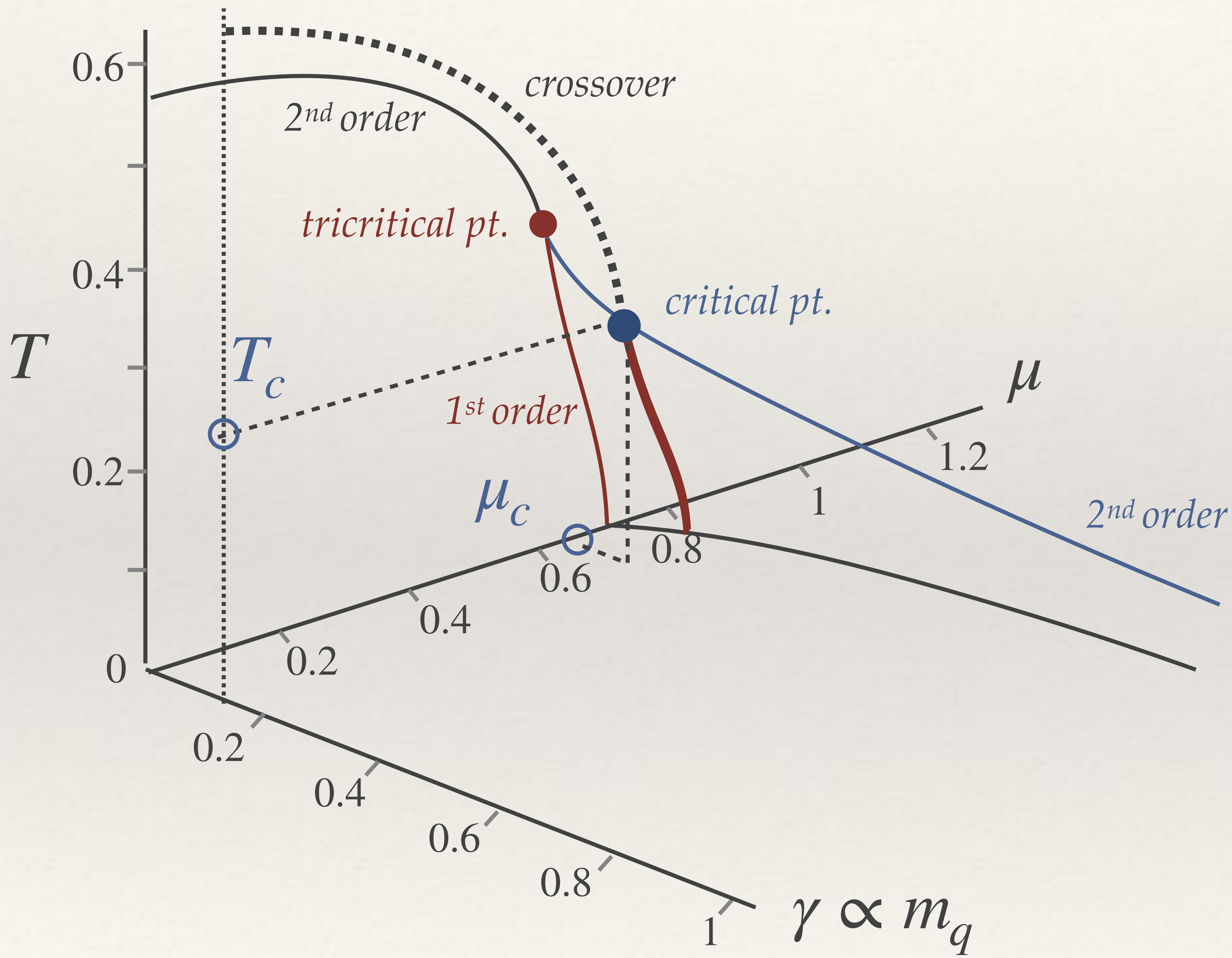
Thermodynamics:

$$\Omega[T,\mu] = \min_{\phi}\left(\frac{\phi^2}{2\pi}\left(\log\phi - \frac{1}{2} + \gamma\right) - \frac{\gamma}{\pi}\phi - T\int\frac{dk}{2\pi}\log\left[\left(1 + e^{-(\sqrt{k^2+\phi^2}-\mu)/T}\right)\left(1 + e^{-(\sqrt{k^2+\phi^2}+\mu)/T}\right)\right]\right)$$

Gross-Neveu Model

"homogeneous" phase diagram (toy example for QCD)*

[Barducci et al. '95]



- Assume $m_q \neq 0$
- Near the critical p.t
  $\Rightarrow \mathbb{Z}_2$ Ising e.o.s

  mean field exponents
  $\beta = 1/2, \ \delta = 3, \ \sigma_{LY} = 1/2$
- Focus on the crossover

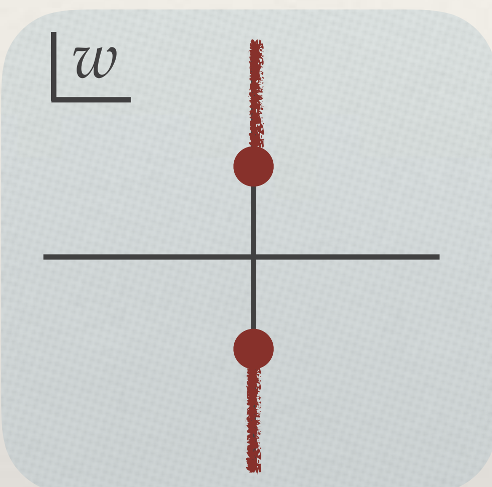
  $$T \gtrsim T_c$$
- T, μ normalized to m

*for the phase diagram including crystalline phases see [Schnetz ,Thies, Ulrichs '05 ; GB, Dunne, Thies '08]

Lee Yang edge singularity

- The scaling e.o.s, $f_s(w)$, has singularities at $w = \pm i w_{LY}$ $(w := hr^{-\beta\delta})$

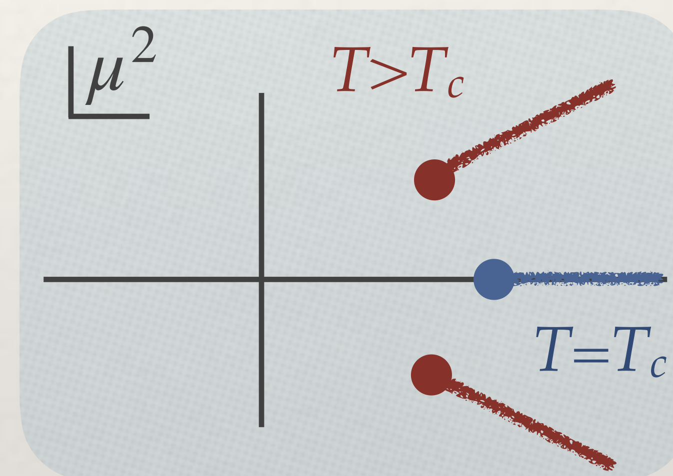
$$\mu_{LY} \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm i w_{LY} \frac{\left(\det M\right)^{\beta\delta}}{h_\mu^{\beta\delta+1}}(T - T_c)^{\beta\delta}$$



slope of the crossover line $(\tan\alpha_1)^{-1}$

$\det M \propto (\tan\alpha_1 - \tan\alpha_2)$

see [Pradeep, Stephanov '19]



- The edge singularities pinch the real axis for $T=T_c$

- There is an associated critical exponent, σ, $M(w) \sim (w \pm i w_{LY})^{\sigma_{LY}}$, $(M$: magnetization)

$$\sigma_{LY,d=3} \approx 0.1, \quad \sigma_{LY,d=6} = 1/2 \text{ (mean field)}$$

see talk by Skokov after this one

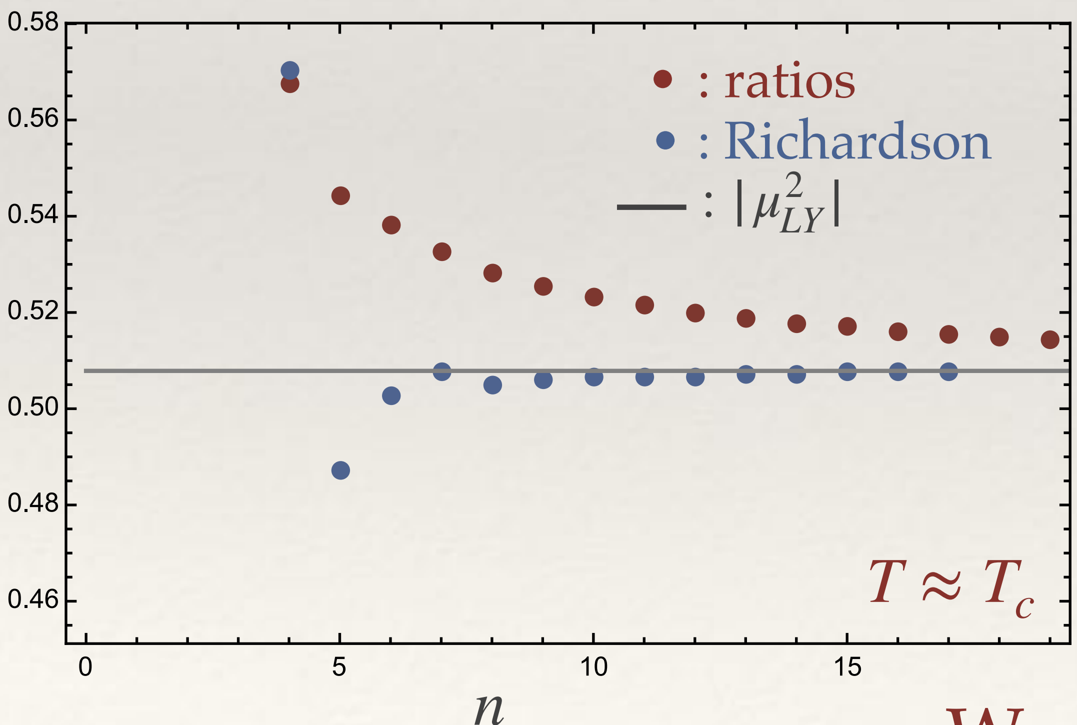
[Fisher, '74; An, Stephanov, Mesterházy '16; Connelly, Johnson, Mukherjee, Skokov '20]

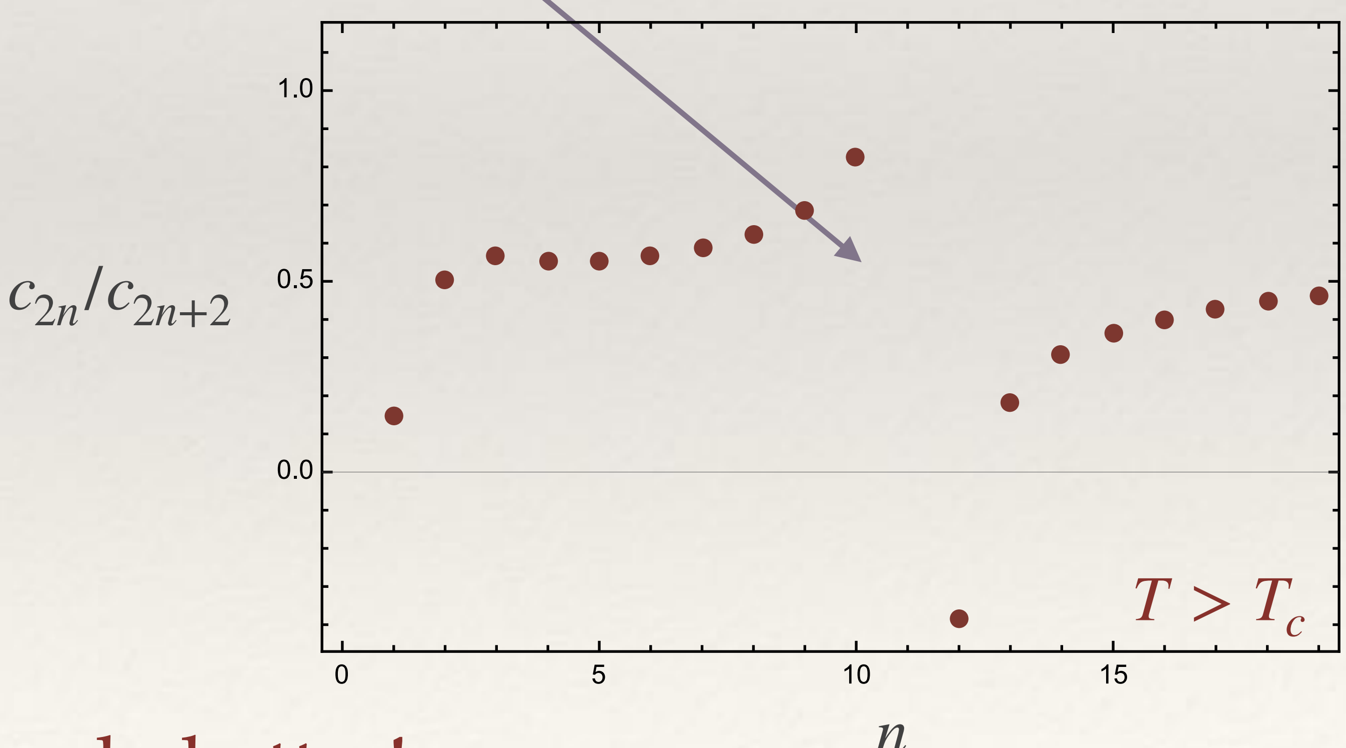
Taylor series

Large order behavior e.g. $\chi(\mu^2) = \sum_n c_{2n}(T)\mu^{2n} \to c_{2n}/c_{2n+2} \sim |\mu_{LY}^2|, \quad n \to \infty$

- In the crossover region the Lee-Yang edge singularity is closest to the origin
- The imaginary part of μ_{LY} could make the large order analysis tricky

$$c_{2n} \sim \frac{\Gamma(\sigma + n)}{\Gamma(1+n)\,|\mu_{LY}^2|^{n+\sigma}} \cos(\theta(n+\sigma) - \pi\sigma), \quad (\theta := \arg \mu_{LY}^2)$$





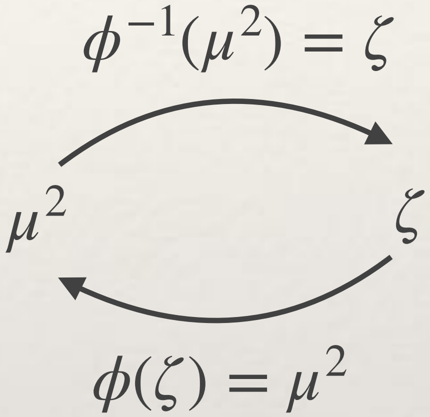
We can do better!

When life gives you Taylor series...

Taylor series: $f(\mu^2) = \sum_{n=0}^{N} c_{2n}\mu^{2n}$

Padé approximant (diagonal) $\mathrm{P}_{[N/2,N/2]}f(\mu^2) = \dfrac{P_{N/2}(\mu^2)}{Q_{N/2}(\mu^2)}$

Padé approximant + conformal mapping: "conformal Padé " $\mathrm{P}f(\phi(\zeta)) = \left.\dfrac{\tilde{P}_{N/2}(\zeta)}{\tilde{Q}_{N/2}(\zeta)}\right|_{\zeta=\phi^{-1}(\mu^2)}$



e.g. $f(x) = \dfrac{1}{\sqrt{1-x}} \approx 1 + \dfrac{x}{2} + \dfrac{5x^3}{16} + \dfrac{3x^4}{8} + \ldots$ $\mathrm{P}_{[3,3]}f(x) = \dfrac{320 + 272x + 2136x^2 - 593x^3}{320 + 112x + 1960x^2 - 1715x^3}$

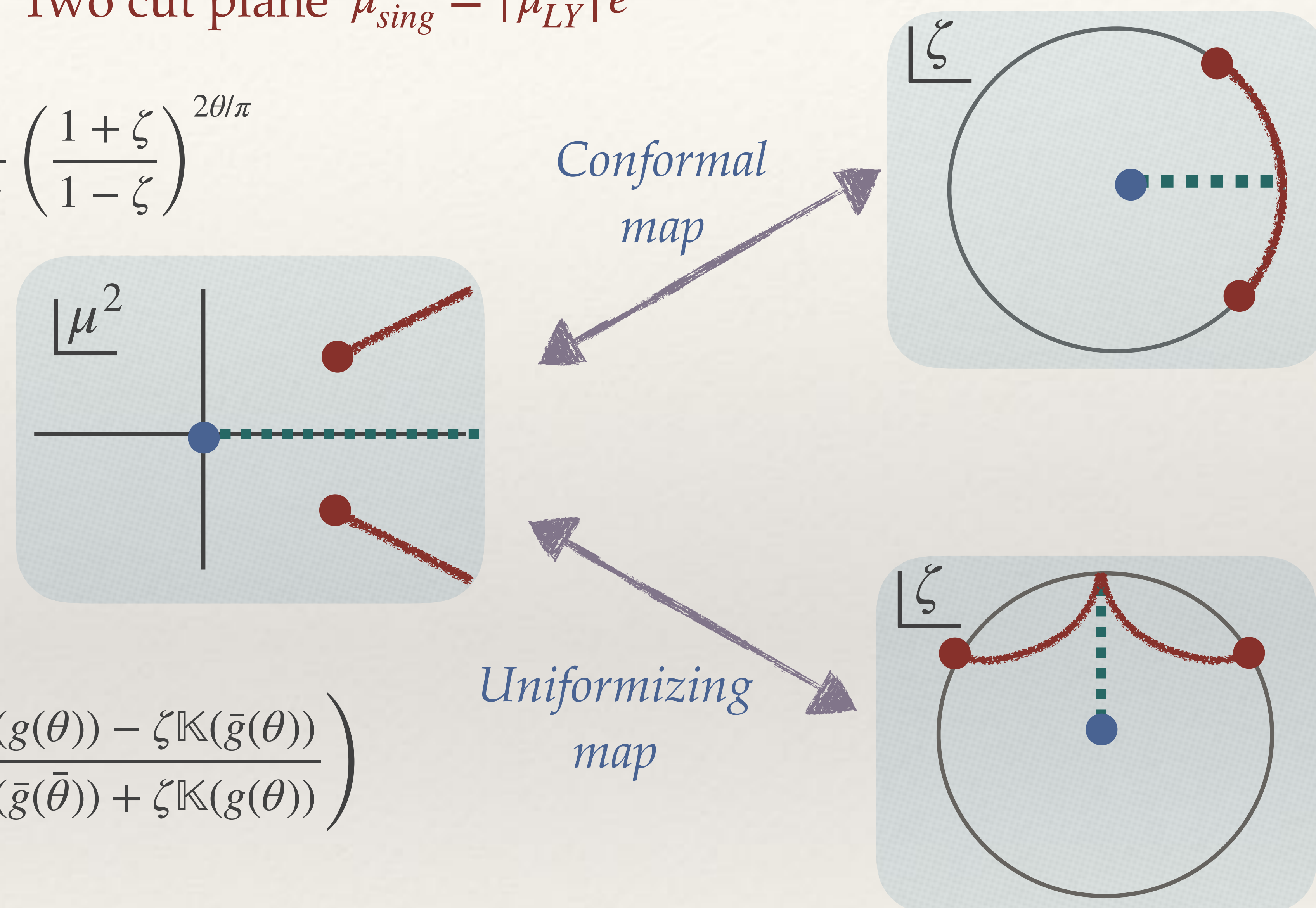
$\phi(\zeta) = \dfrac{4\zeta}{(1+\zeta)^2}$ $\mathrm{P}_{[4,4]}f(\phi(\zeta)) = \dfrac{1+\zeta}{1-\zeta}$ $\phi^{-1}(x) = \dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}$

(used also in Borel summation of ε expansion see eg. [Guida, Zinn-Justin '98]) see talk by Oliviera

Conformal Maps

- Two cut plane $\mu^2_{sing} = |\mu^2_{LY}|\, e^{\pm i\theta}$

$$\phi(\zeta) = \left(\frac{\theta}{\pi}\right)^{\theta/\pi} \left(1 - \frac{\theta}{\pi}\right)^{1-\theta/\pi} \frac{4\mu^2_{LY}\zeta}{(1+\zeta)^2} \left(\frac{1+\zeta}{1-\zeta}\right)^{2\theta/\pi}$$



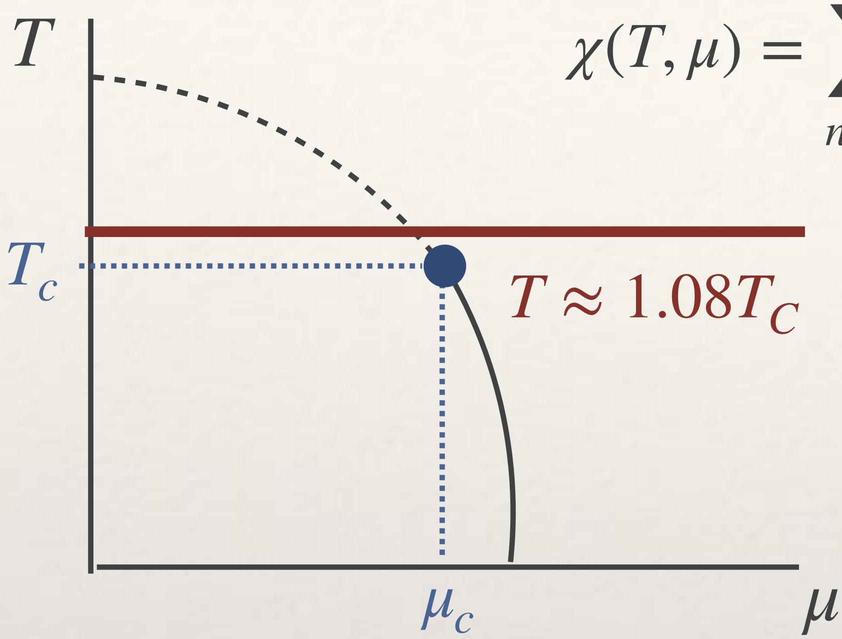
$$\phi(\zeta) = |\mu^2_{LY}|\, e^{i\theta} - 2i\, |\mu^2_{LY}| \sin\theta\, \lambda \left(i\, \frac{\mathbb{K}(g(\theta)) - \zeta \mathbb{K}(\bar{g}(\theta))}{\mathbb{K}(\bar{g}(\bar{\theta})) + \zeta \mathbb{K}(g(\theta))} \right)$$

$$g(\theta) := \frac{1}{2} + \frac{i}{2} \cot\theta$$

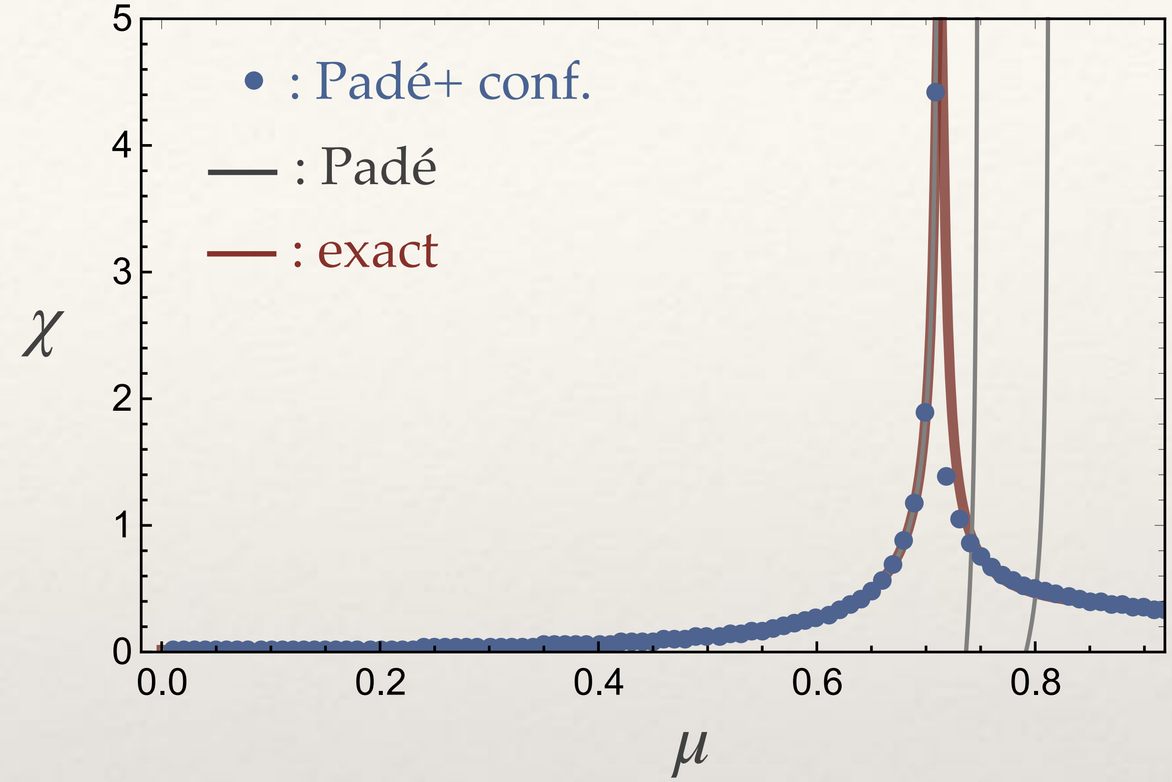
- One cut plane $\mu^2_{sing} = \mu^2_{LY}$
  "local analysis" around each LY singularity $\quad \phi(\zeta) = \dfrac{4\mu^2_{LY}\zeta}{(1+\zeta)^2}$

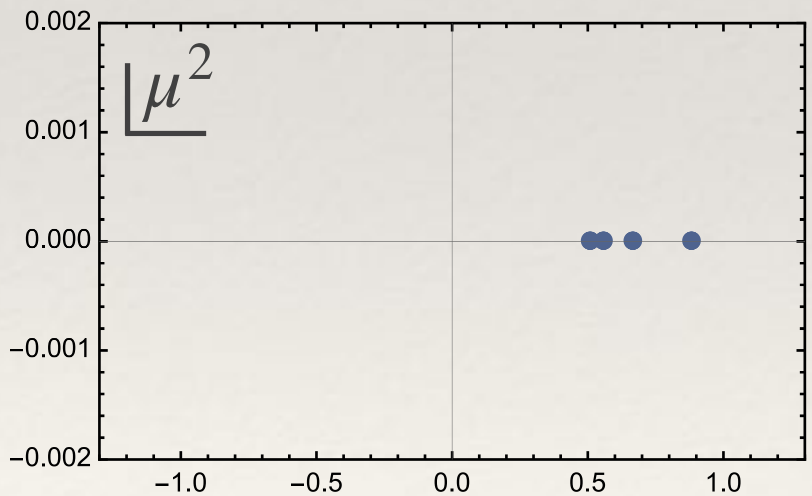
[Costin, Dunne '20]

Results: susceptibility



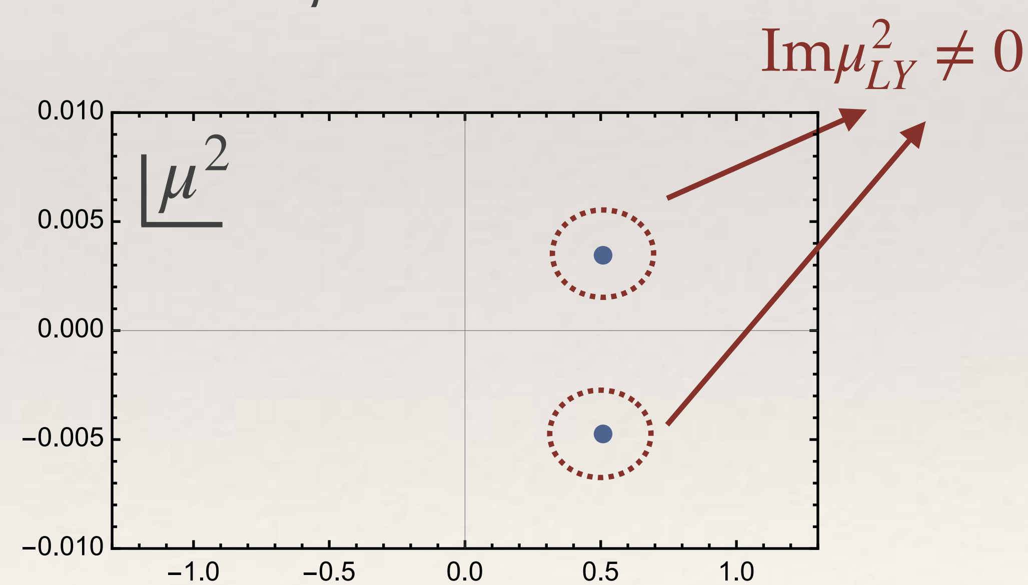
$$\chi(T,\mu) = \sum_{n=0}^{20} c_{2n}(T)\mu^{2n}$$





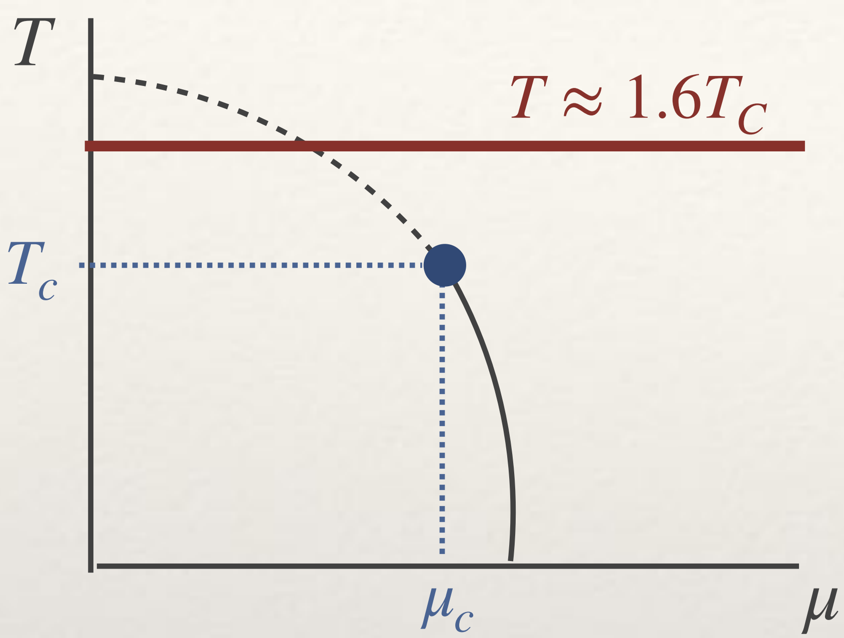
Padé , poles

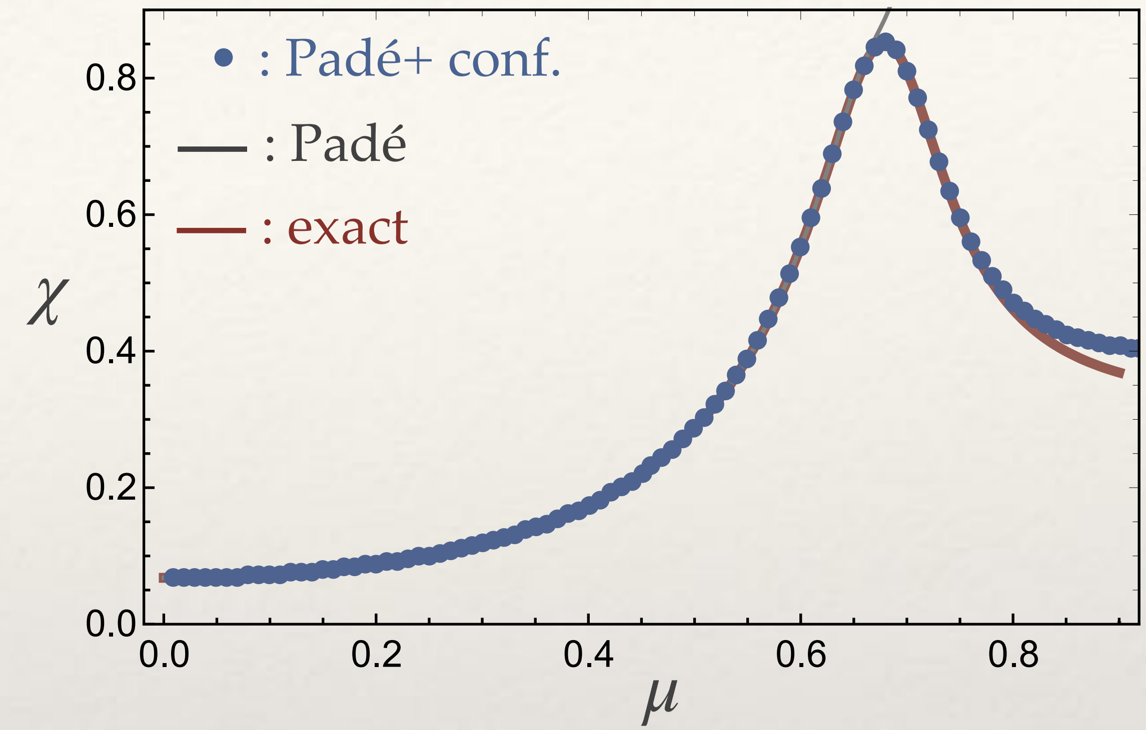
doesn't see $\text{Im}\mu_{LY}^2$

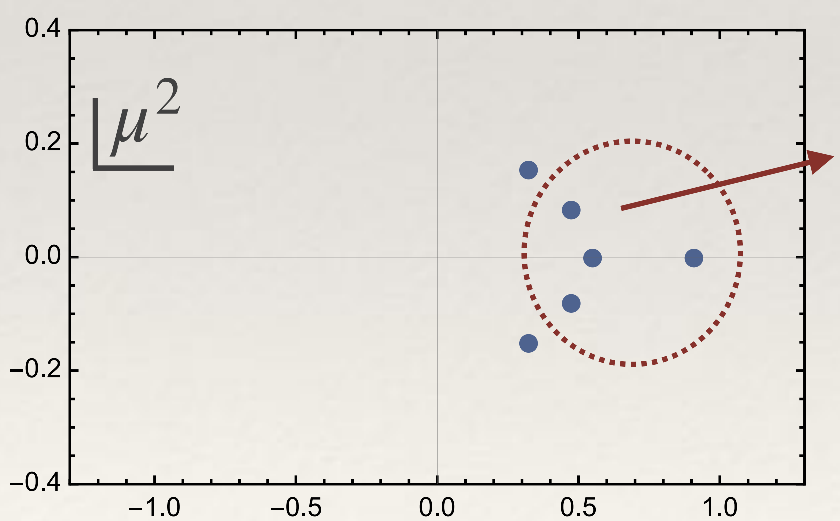


Conformal Padé , poles

Results: susceptibility

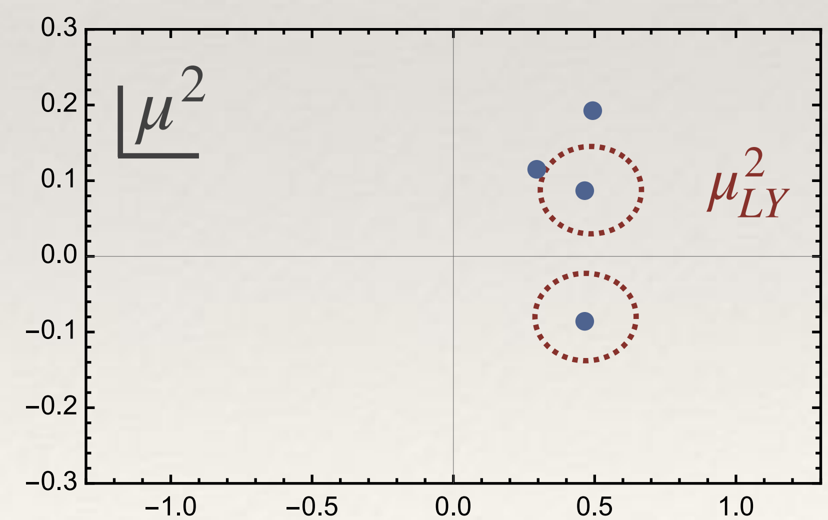






Spurious arcs, limit range of Padé

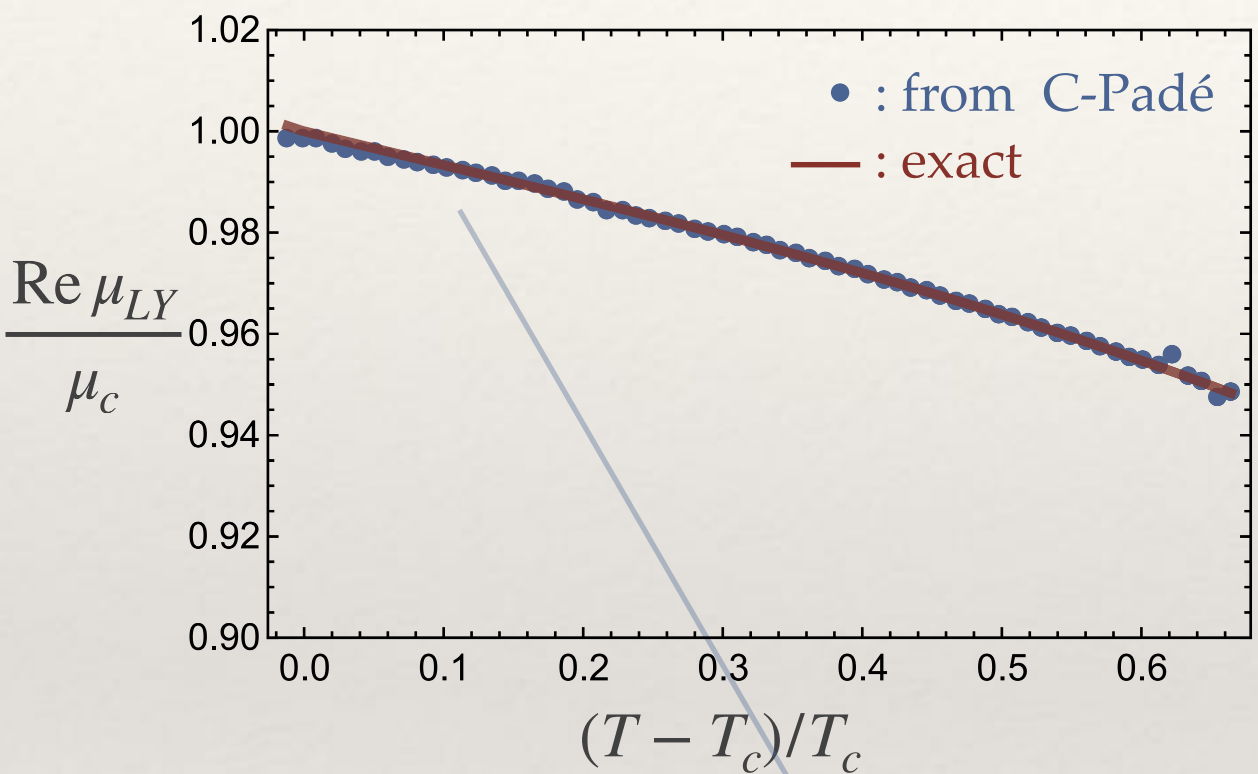
Padé , poles

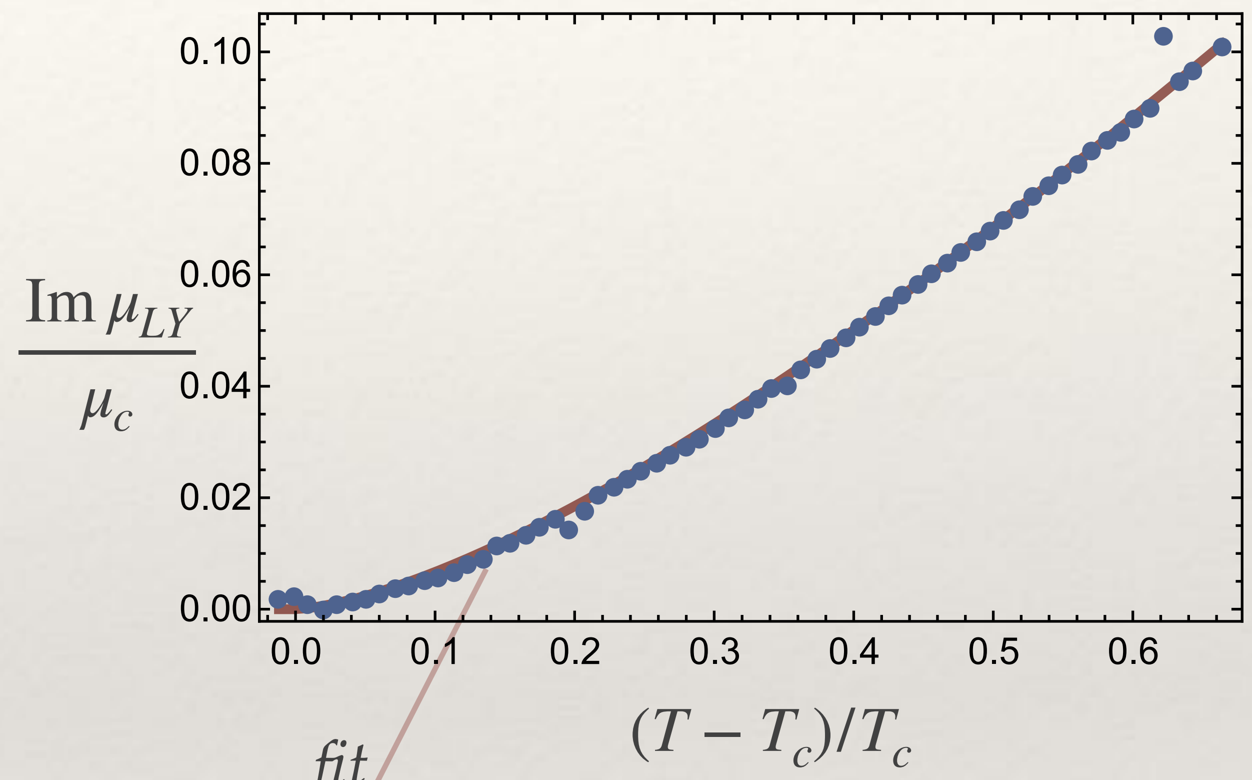


Conformal Padé , poles

Results

- Find $\mu_{LY}^2(T)$ from poles of the conformal- Padé approximants





fit *fit*

$$\mu_{LY} \approx \mu_c - \frac{h_T}{h_\mu}(T-T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}(T-T_c)^{3/2} \qquad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract $\mu_c, T_c,$ crossover slope, $\frac{h_T}{h_\mu}$, and $\frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}$

Conclusions and Outlook

- Combined with conformal maps, Padé approximants provide a powerful tool to extract information from truncated Taylor series.
- In the crossover region by using this tool it is possible to pin down the location of the *Lee-Yang edge singularity* and also extract information on the *mapping parameters to critical Ising e.o.s.*
- Conformal Padé gives a significantly better approximation to the e.o.s than than the Taylor series.
- All of this comes with no more cost than the Taylor series!
- 1st order region
- Singularity elimination
- Crystalline phases?